QUESTION PAPER SOLUTION

MODULE - 1
DIFFERENTIAL EQUATIONS – I

1) $Solve\ 4\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} - 23\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 36 = 0$

(July 2015)

$Sol: The\ given\ equation\ is\ written\ as$

$4D^4 - 4D^3 - 23D^2 + 12D + 36 = 0$

$A.E\ is\ 4m^4 - 4m^3 - 23m^2 + 12m + 36 = 0$

$solving\ we\ get\ m = 2,\ 2,\ 3/2,\ 3/2$

$\therefore\ y_c = c_1 + c_2x\ e^{2x} + c_3 + c_4x\ e^{3/2}$

2) $Solve\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = e^x + 1$ **July 2015**

$Sol: The\ given\ equation\ is\ written\ as\ D^3 + 6D^2 + 11D + 6\ \ y = 0$

$A.E\ is\ m^3 + 6m^2 + 11m + 6 = 0$

$m^3 + m^2 + 5m^2 + 5m + 6m + 6 = 0$

$m^2\ \ m + 1\ \ + 5m(m+1) + 6(m+1) = 0$

$m^2 + 5m + 6 = 0,\ m + 1 = 0$

$\therefore\ m = -1, -2, -3$

$\therefore\ y_c = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}$

$$y_p = \frac{e^x + 1}{D^3 + 6D^2 + 11D + 6} = \frac{e^x}{D^3 + 6D^2 + 11D + 6} + \frac{e^{0x}.1}{D^3 + 6D^2 + 11D + 6}$$

$$= \frac{e^x}{24} + \frac{1}{6}$$

$$GS\ is\ y = \left(c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}\right)\frac{e^x}{24} + \frac{1}{6}$$

3) *Solve* $2\frac{d^2y}{dx^2}-2\frac{dy}{dx}=e^x\sin x$ **July 2015**

Sol : The given equation is written as $2D^2-2D\ y=e^x\sin x$

$A.E\ is\ m^2-2m=0$

$m(m-2)=0$

$\therefore m=0,2$

$\therefore y_c=c_1+c_2e^{2x}$

let PI as $y=e^x(A\cos x+b\sin x)$

Differentiating w.r.t x $y'=e^x(-A\sin x+b\cos x)+e^x(A\cos x+b\sin x)$

$$y''=e^x\{(A+B)\cos x-(A+B)\sin x)\}+e^x\{(-A+B)\sin x+(A+B)\cos x\}$$

Sustituting these values in the given equations and solving we get

$A=0,\ B=-1/2\rightarrow y_p=e^x\left(\frac{-1}{2}\sin x\right)$

$$G\,S\ \ is\ y=y_c+y_p=c_1+c_2e^{2x}-e^x\left(\frac{1}{2}\sin x\right)$$

4) *Solve* : $y''+4y'-12y=e^{2x}-3\sin 2x$ Jan 2016

Sol : We have $(D^2+4D-12)y=e^{2x}-3\sin 2x$

$$AE: m^2+4m-12=0\Rightarrow m=2,\ -6$$

$$y_c=c_1e^{2x}+c_2e^{-6x}$$

$$y_p=\frac{e^{2x}}{D^2+4D-12}-\frac{3\sin 2x}{D^2+4D-12}$$

$$=x\frac{e^{2x}}{D+4}-\frac{3\sin 2x}{4\ D-4}$$

$$=x\frac{e^{2x}}{8}-\frac{3\ D+4\ \sin 2x}{4\ D^2-16}$$

$$=\frac{xe^{2x}}{8}-\frac{3\ 2\cos x+4\sin 2x}{80}$$

General solution

$$y=y_c+y_p\qquad \therefore y=c_1e^{2x}+c_2e^{-6x}+\frac{xe^{2x}}{8}-\frac{3\ 2\cos x+4\sin 2x}{80}$$

5) By the method of undetermined coefficients solve $y'' + y = 2\cos x$ **(Jan 2016)**

Sol: $we\ have\ (D^2+1)y = 2\cos x$

$$AE: m^2+1=0 \Rightarrow m = \pm i$$
$$y_c = c_1\cos x + c_2 \sin x$$
$$Assume\ \ y_p = x(A\cos x + B\sin x)$$
$$y_p' = x(-A\sin x + B\cos x) + A\cos x + B\sin x$$
$$y_p'' = x(-A\cos x - B\sin x) - 2A\sin x + 2B\cos x$$
$$substituting\ in\ given\ equation\ and\ comparing\ coefficients\ we\ get$$
$$B = 1\ and\ \ A = 0$$
$$y_p = x\sin x$$
$$G.S\ is\ given\ by$$
$$y = c_1\cos x + c_2\sin x + x\sin x$$

6) By the method of variation of parameters solve $y'' + 4y = \tan 2x$ **(Jan 2016)**

Sol: $(D^2+4)y = \tan 2x$

$$AE: m^2+4=0 \Rightarrow m = \pm 2i$$
$$y_c = c_1\cos 2x + c_2\sin 2x$$
$$y = A\cos 2x + B\sin 2x$$
$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$
$$A = \int \frac{\sin 2x \tan 2x}{2} dx = \frac{\sin 2x}{4} - \frac{\log(\sec 2x + \tan 2x)}{4} + k_1$$
$$B = \int \frac{\cos 2x \tan 2x}{2} dx = \frac{-\cos 2x}{4} + k_2$$
$$G.S\ is\ given\ by$$
$$y = \left(\frac{\sin 2x}{4} - \frac{\log(\sec 2x + \tan 2x)}{4} + k_1\right)\cos 2x + \left(\frac{-\cos 2x}{4} + k_2\right)\sin 2x$$

7) $Solve\ (D^4+m^4)y=0$ (Jan 2016)

$solution:\ A.E\quad D^2\ ^2+\ m^2\ ^2=0$

$$(D^2+m^2)^2-2D^2m^2=0$$

$$D^2+m^2+\sqrt{2}Dm\quad D^2+m^2-\sqrt{2}Dm\ =0$$

$$D=\frac{\sqrt{2}a\pm\sqrt{2}ai}{2},\quad D=\frac{-\sqrt{2}a\pm\sqrt{2}ai}{2}$$

$$=\frac{a}{\sqrt{2}}\pm\frac{a}{\sqrt{2}}i\qquad =-\frac{a}{\sqrt{2}}\pm\frac{a}{\sqrt{2}}i$$

$\therefore$ *The general solution is given by*

$$y=e^{\frac{ax}{\sqrt{2}}}\left(c_1\cos\frac{a}{\sqrt{2}}x+c_2\sin\frac{a}{\sqrt{2}}x\right)+e^{\frac{-ax}{\sqrt{2}}}\left(c_1\cos\frac{a}{\sqrt{2}}x+c_2\sin\frac{a}{\sqrt{2}}x\right)$$

8) $Solve\ (D^4+m^4)y=0$ (Jan 2016)

$Sol:\ A.E\quad D^2\ ^2+\ m^2\ ^2=0$

$$(D^2+m^2)^2-2D^2m^2=0$$

$$D^2+m^2+\sqrt{2}Dm\quad D^2+m^2-\sqrt{2}Dm\ =0$$

$$D=\frac{\sqrt{2}a\pm\sqrt{2}ai}{2},\quad D=\frac{-\sqrt{2}a\pm\sqrt{2}ai}{2}$$

$$=\frac{a}{\sqrt{2}}\pm\frac{a}{\sqrt{2}}i\qquad =-\frac{a}{\sqrt{2}}\pm\frac{a}{\sqrt{2}}i$$

$\therefore$ The general solution is given by

$$y=e^{\frac{ax}{\sqrt{2}}}\left(c_1\cos\frac{a}{\sqrt{2}}x+c_2\sin\frac{a}{\sqrt{2}}x\right)+e^{\frac{-ax}{\sqrt{2}}}\left(c_1\cos\frac{a}{\sqrt{2}}x+c_2\sin\frac{a}{\sqrt{2}}x\right)$$

9) Solve $(D^2+7D+12)y=\cosh x$ **(Jan 2016)**

Sol: We have $(D^2+7D+12)y=\dfrac{e^x+e^{-x}}{2}$

$$AE: m^2+7m+12=0\Rightarrow m=-3,\ -4$$

$$y_c=c_1e^{-3x}+c_2e^{-4x}$$

$$y_p = \frac{1}{2}\left(\frac{e^x}{D^2+7D+12} + \frac{e^{-x}}{D^2+7D+12}\right)$$

$$= \frac{1}{2}\left(\frac{e^x}{20} + \frac{e^{-x}}{6}\right)$$

General solution

$$y = y_c + y_p$$

$$y = c_1 e^{-3x} + c_2 e^{-4x} + \frac{1}{2}\left(\frac{e^x}{20} + \frac{e^{-x}}{6}\right)$$

10) By the method of variation of parameters solve $y'' + y = x\sin x$ **(Jan 2016)**

Sol : $D^2+1 \ \ y = x\sin x$

$AE : m^2 + 1 = 0 \Rightarrow m = \pm i$

$y_c = c_1\cos x + c_2 \sin x$

$y = A\cos x + B\sin x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$A = \int \frac{x\sin^2 x}{2} dx = \frac{-1}{2}\left(\frac{x^2}{2} - \frac{x\sin x}{2} - \frac{\cos 2x}{4}\right) + k_1$$

$$B = \int x\sin x\cos x dx = \frac{-x\cos 2x}{4} + \frac{\sin 2x}{8} + k_2$$

G.S is given by

$$y = \left(\frac{-1}{2}\left(\frac{x^2}{2} - \frac{x\sin x}{2} - \frac{\cos 2x}{4}\right) + k_1\right)\cos x + \left(\frac{-x\cos 2x}{4} + \frac{\sin 2x}{8} + k_2\right)\sin x$$

11) $Solve\,(4D^4 - 8D^3 - 7D^2 + 11D + 6)y = 0$ **(June 2015)**

Solution: *The AE is* $4m^4 - 8m^3 - 7m^2 + 11m + 6 = 0$

By inspection method m=-1 and m=2 are two roots

then we get $4m^2 - 4m - 3 = 0 \Rightarrow m = \frac{-1}{2}, \frac{3}{2}$

$\therefore\ y = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{-\frac{1}{2}x} + c_1 e^{\frac{3}{2}x}$

12) $Solve\,(D^2+4)y = x^2+e^{-x}$ **(June 2015)**

Solution: $The\ AE\ is\ m^2+4=0 \Rightarrow m=\pm 2i$

$$\therefore y_c = c_1\cos 2x + c_2 \sin 2x$$

$$y_p = \frac{1}{D^2+4}x^2 + \frac{1}{D^2+4}e^{-x}$$

$$= \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}x^2 + \frac{e^{-x}}{5}$$

$$= \frac{1}{4}\left(1-\frac{D^2}{4}+\left(\frac{D^2}{4}\right)^2 - \ldots\ldots\ldots\right)x^2 + \frac{e^{-x}}{5}$$

$$= \frac{1}{4}\left(x^2 - \frac{1}{4}.2+0\right) + \frac{e^{-x}}{5} = \frac{1}{4}\left(x^2-\frac{1}{2}\right)+\frac{e^{-x}}{5}$$

$$\therefore y = c_1\cos 2x + c_2\sin 2x + \frac{1}{4}\left(x^2-\frac{1}{2}\right)+\frac{e^{-x}}{5}$$

13) $Solve\,(D^2-2D+2)y = e^x \tan x$ using $method\ of\ variation\ of\ parameters.$ **(June 2015)**

Solution: $The\ AE\ is\ m^2-2m+2=0 \Rightarrow m = 1\pm i$

$$\therefore y_c = e^x\ (c_1\cos x + c_2 \sin x)$$

$$u = e^x\cos x \qquad v = e^x \sin x$$

$$w = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^x\cos x & e^x \sin x \\ e^x(-\sin x+\cos x) & e^x(\sin x + \cos x) \end{vmatrix} = e^{2x}$$

$$A = -\int \frac{vf(x)}{w}dx = -\int\frac{e^x\sin x.e^x\tan x}{e^{2x}}dx = -\int\frac{\sin^2 x}{\cos x}dx$$

$$= -\int (\sec x - \cos x)\ dx = \sin x - \log(\sec x + \tan x) + k_1$$

$$B = \int\frac{uf(x)}{w}dx = \int\frac{e^x\cos x.e^x\tan x}{e^{2x}}dx = \int \sin x dx = -\cos x + k_2$$

$Hence\ general\ slolution\ is$

$$y = e^x\left(A\cos x + B\sin x\right)$$

$$y = e^x\left(c_1\cos x + c_2\sin x\right) - e^x\cos x\log\left(\sec x+\tan x\right)$$

14) Solve $(D^3 - D)y = 2e^x + 4\cos x$ **(Jan 2015)**

Solution: $Given\,(D^3 - D)y = 2e^x + 4\cos x$

$$A.E\ is\ (D^3 - D)y = 0$$

$$\Rightarrow m^3 - m = 0$$

$$m(m^3 - 1) = 0$$

$$\Rightarrow m = 0,\ \ m^2 - 1 = 0 \qquad \Rightarrow m = I1$$

$$\Rightarrow m = 0, -1, 1$$

$$C.F = c_1 e^{ox} + c_2 e^{-x} + c_3 e^x$$

$$C.F = c_1 + c_2 e^{-x} + c_3 e^x$$

$$P.I = \frac{2e^x + 4\cos x}{f(D)} = \frac{2e^x + 4\cos x}{D^3 - D}$$

$$= \frac{2e^x}{D^3 - D} + \frac{4\cos x}{D^3 - D}$$

$$= \frac{2e^x}{1-1} + \frac{4\cos x}{-D - D}$$

$$= \frac{2xe^x}{3D^2 - 1} + \frac{4\cos x}{-2D}$$

$$= \frac{2xe^x}{3.1 - 1} - \frac{2\cos x}{D}$$

$$= \frac{2xe^x}{2} - 2\sin x$$

$$P.I = xe^x - 2\sin x$$

The solution is y = C.F. +P.F

$$y = c_1 + c_2 e^{-x} + c_3 e^x + xe^x - 2\sin x$$

15) Solve: ($D^2 + 2$ **)y=** $x^2 e^{3x} + e^x \cos 2x$ **(Jan 2015)**

Solution: $L(D) = D^2 + 2,$ *which leads to*

$$C.F. = A\cos\sqrt{2}x + B\sin\sqrt{2}x.$$

Also,

$$P.I. = \frac{1}{L(D)}\ \ x^2 e^{3x} + e^x \cos 2x \ = \frac{1}{L(D)} e^{3x} x^2 + \frac{1}{L(D)} e^x \cos 2x$$

$$= e^{3x}\frac{1}{L(D+3)}x^2 + e^{3x}\frac{1}{L(D+1)}\cos 2x$$

$$= e^{3x}\frac{1}{(D+3)^2+2}x^2 + e^{x}\frac{1}{(D+1)^2+2}\cos 2x$$

$$= e^{3x}\frac{1}{D^2+6D+11}x^2 + e^{x}\frac{1}{D^2+2D+3}\cos 2x$$

$$= e^{3x}\frac{1}{11}\left\{1+\frac{1}{11}\left(D^2+6D\right)\right\}^{-1}x^2 + e^{x}\frac{1}{-2^2+2D+3}\cos 2x$$

$$= \frac{1}{11}e^{3x}\left\{1-\frac{1}{11}\left(D^2+6D\right)+\frac{1}{11^2}\left(D^2+6D\right)^2 \ldots\ldots\right\}x^2 + e^{x}\frac{1}{2D-1}\cos 2x$$

$$= \frac{1}{11}e^{3x}\left\{1-\frac{1}{11}\left(D^2+6D\right)+\frac{1}{11^2}\left(D^4+12D^3+36D^2\right)\right\}x^2 + e^{x}\frac{2D-1}{4D^4-1}\cos 2x$$

$$= \frac{1}{11}e^{3x}\left\{x^2-\frac{1}{11}\left(2+12x\right)+\frac{1}{11^2}\left(72\right)\right\} + e^{x}\frac{2D+1}{4(-2^2)-1}\left(\cos 2x\right)$$

$$= \frac{1}{11}e^{3x}\left(x^2-\frac{12}{11}x+\frac{50}{121}\right)-\frac{1}{17}e^{x}(-4\sin 2x+\cos 2x)$$

Therefore, the general solution of the given equation is

Y=C.F.+P.I.=$A\cos\sqrt{2x}+B\sin\sqrt{2x}+\frac{1}{11}e^{3x}\left(x^2-\frac{12}{11}x+\frac{50}{121}\right)+\frac{1}{17}e^{x}(4\sin 2x-\cos 2x)$.

16) Solve the simultaneous equation (D+5) x-2y=t and (D+1) y+2x=0 (Jan2015)

Solution: The given equations are

(D+5) x-2y=t

(D+1) y+2x=0

From equation (ii), we get

(D+5) (D+1)y +2(D+5)x=0.

Using equation (i), this becomes

(D+5) (D+1)y +2(t+2y)=0.

For this equation, the A.E. is $(m+3)^2 = 0$ whose toots are -3, -3. Therefore

C.F. = $(c_1+ c_2t)\, e^{-3t}$

$$P.I.=\frac{1}{(D+3)^2}(-2t)=\frac{1}{9}\frac{1}{(1+D/3)^2}(-2t)$$

$$=-\frac{2}{9}\left(1+\frac{D}{3}\right)^{-2}(t)=-\frac{2}{9}\left(1-\frac{2}{3}D+\ldots\ldots\right)t=-\frac{2}{9}\left(t-\frac{2}{3}\right)$$

Therefore, the general solution of equation (iii) is

Y = C.F. + P.I. = $(c_1+ c_2t)e^{-3t}-\frac{2}{9}t+\frac{4}{27}$

Using this in equation (ii), we get

$$2x = -(D+1)y = -(D+1)\left((c_1 + c_2) e^{-3t} - \frac{2}{9}t + \frac{4}{27} \right)$$

$$= -\left[\left\{ (-3)(c_1 + c_2) e^{-3t} + c_2 e^{-3t} - \frac{2}{9} \right\} + \left\{ (c_1 + c_2 t) e^{-3t} - \frac{2}{9}t + \frac{4}{27} \right\} \right]$$

$$= -\left[(-2)(c_1 + c_2) e^{-3t} + c_2 e^{-3t} - \frac{2}{9}t - \frac{2}{27} \right]$$

$$= (2c_1 - c_2 + 2ct) e^{-3t} - \frac{2}{9}t - \frac{2}{27}$$

$$x = \left\{ \left(c_1 - \frac{1}{2}c_2 \right) + c_2 t \right\} e^{-3t} - \frac{2}{9}t - \frac{1}{27}$$

Expressions (iv) and (v) constitute the general solution of the given system.
It is given that x=0 and y=0 when t = 0. Using this condition in expressions (iv) and (v), we get.

$$0 = c_1 + \frac{4}{27}, \quad 0 = \left(c_1 - \frac{1}{2}c_2 \right) + \frac{1}{27}$$

From these, we get

$$c_1 = -\frac{4}{27}, \; c_2 = 2\left(c_1 + \frac{1}{27} \right) = 2\left(-\frac{3}{27} \right) = -\frac{2}{9}$$

Substituting these values of c_1 and c_2 into expressions (V) and (iV), we get

$$x = -\frac{1}{27}(1+6t)e^{-3t} + \frac{1}{27}(3t+1)$$

$$y = -\frac{1}{27}(2+3t)e^{-3t} - \frac{1}{27}(3t-2)$$

These constitute the required solution of the given system.

17) Solve $(D-2)^2 y = 8(e^{2x} + \sin 2x)$ **(June 2014)**

Solution: $A.E \text{ is } (m-2)^2 = 0$

$$\Rightarrow m = 2, 2,$$

$$C.F = (c_1 + c_2 x) e^{2x}$$

$$P.I = \frac{8(e^{2x} + \sin 2x)}{f(D)} = 8\frac{e^{2x} + \sin 2x}{(D-2)^2} = 8\left[\frac{e^{2x}}{(D-2)^2} + \frac{\sin 2x}{D^2 - 4D + 4} \right]$$

$$= 8\left[\frac{xe^{2x}}{2(D-2)} + \frac{\sin 2x}{-4D} \right] = 8\left[\frac{x^2 e^{2x}}{2} + \frac{\cos 2x}{8} \right]$$

$$= 4x^2 e^{2x} + \cos 2x$$

The solution is y = C.F. +P.F

$$y = (c_1 + c_2 x) e^{2x} + 4x^2 e^{2x} + \cos 2x$$

18) Solve: $y'' - 2y' + y = x\cos x$ **(June 2014)**

Solution:

$$A.E \text{ is } m^2 - 2m + 1 = 0$$

$$\Rightarrow m = 1,\ 1$$

$$C.F = \ c_1 + c_2 x \ \ e^x$$

$$P.I = \frac{x\cos x}{D^2 - 2D + 1} = \left[x - \frac{2D - 2}{D^2 - 2D + 1} \right] \frac{\cos x}{D^2 - 2D + 1}$$

$$= \left[x - \frac{2\ \ D - 1}{D - 1\ \ ^2} \right] \frac{\cos x}{-2D} = \left[x - \frac{2}{D - 1} \right] \left(\frac{\sin x}{-2} \right)$$

$$= \frac{-x\sin x}{2} + \frac{\sin x}{D - 1} = \frac{-x\sin x}{2} + \frac{(D + 1)\sin x}{D^2 - 1}$$

$$= \frac{-x\sin x}{2} + \frac{\cos x + \sin x}{-2} = \frac{-1}{2}\ \ x\sin x + \cos x + \sin x$$

The solution is y = C.F. +P.F

$$y = \ \ c_1 + c_2 x \ \ e^{2x} + \frac{-1}{2}\ \ x\sin x + \cos x + \sin x$$

19) Solve $\dfrac{dx}{dt} - 7x + y = 0,\ \dfrac{dy}{dt} - 2x - 5y = 0$ **(June 2014)**

Sol : The given equations can be written as

$$D - 7\ \ x + y = 0.............(1)$$

$$-2x + \ D - 5\ \ y = 0..............(2)$$

Operating (2) by (D-7) and multiplying (1) by 2 we have

$$2\ \ D - 7\ \ x + 2y = 0$$

$$-2\ \ D - 7\ \ x + \ D - 5\ \ \ D - 7\ \ y = 0$$

$$Adding\ we\ get, \left[\ D - 5\ \ \ D - 7\ \ + 2 \right] y = 0$$

$$or\ \ \ D^2 - 12D + 37\ \ y = 0$$

$$A.E\ is\ \ \ m^2 - 12m + 37 = 0 \Rightarrow m = 6 \pm i$$

$$\therefore y = e^{6t}\ \ c_1\cos t + c_2\sin t\ \ ....(3)$$

By considering $\dfrac{dy}{dt} - 2x - 5y = 0$ we get, $x = \dfrac{1}{2}\left[\dfrac{dy}{dt} - 5y \right]$

$$x = \frac{1}{2}\left[\frac{d}{dt}\ \ e^{6t}\ \ c_1\cos t + c_2\sin t\ \ \ - 5e^{6t}\ \ c_1\cos t + c_2\sin t\ \ \right]$$

$$= \frac{1}{2}\left[e^{6t}\left(-c_1 \sin t + c_2 \cos t\right) + 6e^{6t}\left(c_1 \cos t + c_2 \sin t\right) - 5e^{6t}\left(c_1 \cos t + c_2 \sin t\right)\right]$$

$$\therefore x = \frac{1}{2}\left[\left(c_1 + c_2\right) e^{6t} \cos t + \left(c_2 - c_1\right) e^{6t} \sin t\right] \quad ....(4)$$

(3) and (4) represents the general solution of given system of equations

20) Solve $\frac{dx}{dt} - 2y = \cos 2t,\ \frac{dy}{dt} + 2x = \sin 2t,\ given\, that\ x = 1, y = 0\, at\, t = 0$ **(Dec2013)**

Sol: The given equations can be written as

$$Dx - 2y = \cos 2t ....(1)$$

$$2x + Dy = \sin 2t ....(2)$$

Operating (1) by D and multiplying (2) by 2 we have

$$D^2 x - 2Dy = -2\sin 2t$$

$$4x + 2Dy = 2\sin 2t$$

$$Adding\ we\ get,\quad \left(D^2 + 4\right) x = 0$$

$$A.E\ is \qquad m^2 + 4 = 0$$

$$\therefore x = c_1 \cos 2t + c_2 \sin 2t ....(3)$$

By considering $\frac{dx}{dt} - 2y = \cos 2t$ we get, $y = \frac{1}{2}\left[\frac{dx}{dt} - \cos 2t\right]$

$$y = \frac{1}{2}\left[\frac{d}{dt}\left(c_1 \cos 2t + c_2 \sin 2t\right) - \cos 2t\right]$$

$$= \frac{1}{2}\left[\left(-2c_1 \sin 2t + 2c_2 \cos 2t\right) - \cos 2t\right]$$

$$\therefore y = -c_1 \sin 2t + \left(c_2 - 1/2\right) \cos 2t ....(4)$$

(3) and (4) represents the general solution of given system of equations

It is given that x=1 and y=0 when t = 0. Using this condition in expressions (3) and (4), we get.

$$from\,(3)\, 1 = c_1 + 0 \Rightarrow c_1 = 1$$

$$from\,(4)\, 0 = 0 + \left(c_2 - 1/2\right) \Rightarrow c_2 = 1/2$$

Substituting these values of c_1 and c_2 into expressions (3) and (4), we get

$$x = \cos 2t + \frac{\sin 2t}{2};\ \ y = -\sin 2t$$

21) Using the method of variation of variation of parameters solve $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$.

(June 2014, Dec 2013)

Solution: ; $We\, have\ \ D^2 - 6D + 9\ \ y = \frac{e^{3x}}{x^2}$

$A.E.\quad m^2 - 6m + 9 = 0 \Rightarrow m = 3,3$

$y_c = \ c_1 + c_2 x\ \ e^{3x}$

$y = \ A + Bx\ \ e^{3x}$ be the complete solution of the given equation where A(x),B(x) are to be found

We have $y_1=e^{3x}$ and $y_2=xe^{3x}$

$$A' = \frac{-y_2\varphi\ \ x}{W}\ , \qquad B' = \frac{y_1\varphi(x)}{W}$$

$$A' = \frac{-xe^{3x}.e^{3x}/x^2}{e^{6x}}, \qquad B' = \frac{e^{3x}.e^{3x}/x^2}{e^{6x}}$$

$$A' = \frac{-1}{x}\ , \qquad B' = \frac{1}{x^2}$$

Integrating we get

$$A = -\log x + k_1 \qquad B = \frac{-1}{x}\ \ + k_2$$

Using the expression of A and B in in $y = \ A + Bx\ \ e^{3x}$ we have,

$$y = \ -\log x + k_1\ \ e^{3x} + \left(\frac{-1}{x}\ \ + k_2\right) xe^{3x}$$

$$y = \ k_1 + k_2\, x\ \ e^{3x} - e^{3x}\log x$$

22) Solve: $x^2 y'' + xy' + y = 2\cos^2(\log x)$. **(Dec 2013)**

Solution: Put logx=t or $e^t = x$

$$xy' = Dy,\ x^2 y'' = D(D-1)y \quad where\ D = \frac{d}{dt}$$

$$D(D-1)y - Dy + y\ \ = 2\cos^2 t$$

$$D(D-1) - D + 1\ \ = 2\cos^2 t$$

$$\left[D^2 - 2D + 1\right] y\ = 2\cos^2 t$$

$$m^2 - 2m + 1 = 0$$

$$m = 1$$

$$y_c = (c_1 + c_2 t)e^t$$

$$yp = \frac{2\cos^2 t}{D^2 - 2D + 1}$$

$$= \frac{1 + \cos 2t}{D^2 - 2D + 1}$$

$$= \frac{1}{D^2 - 2D + 1} + \infty \frac{\cos 2t}{D^2 - 2D + 1} = P_1 + P_2$$

$$P_1 = \frac{1}{D^2 - 2D + 1} = \frac{e^{ox}}{D^2 - 2D + 1} = \frac{e^{ox}}{0 - 2(0) + 1} = \frac{1}{1} = 1$$

$$P_2 = \frac{\cos 2t}{D^2 - 2D + 1} = \frac{\cos 2t}{-4 - 2D + 1} = \frac{\cos 2t}{-2D - 3}$$

$$= \frac{(2D - 3)\cos 2t}{-(2D + 3)(2D - 3)}$$

$$= \frac{2D(\cos 2t) - 3\cos 2t}{-(4D^2 - 9)}$$

$$= \frac{-4\sin 2t - 3\cos 2t}{25}$$

$$= \frac{-\ \ 4\sin 2t + 3\cos 2t}{25}$$

MODULE-2

DIFFERENTIAL EQUATIONS - II

1) Solve the simultaneous equations $\frac{dx}{dt}+2y+\sin t=0,\ \frac{dy}{dt}-2x-\cos t=0$ **given that x = 0, y = 1 when t = 0** **(Jan 2016)**

Sol: Given $Dx+2y=-\sin t,\ -2x+Dy=\cos t$

$$\textit{Solving the equations we get } (D^2+4)y=-3\sin t$$
$$\Rightarrow m^2+4=0 \Rightarrow m=\pm 2i$$
$$y_c=c_1\cos 2t+c_2\sin 2t$$
$$y_p=\frac{-3\sin t}{D^2+4}=-\sin t$$
$$\therefore y=c_1\cos 2t+c_2\sin 2t-\sin t$$
$$\textit{Substituting } \frac{dy}{dt} \textit{ in the given equation we get}$$
$$x=-c_1\sin 2t+c_2\cos 2t-\cos t$$

2) Solve $x^2y''-xy'+2y=x\sin(\log x)$ **(Jan 2016)**

Sol: $put\ \ x=e^t \Rightarrow t=\log x$

$$\textit{then we have } (D(D-1)-D+2)y=e^t\sin t$$
$$AE\quad m^2-2m+2=0 \Rightarrow m=1\pm i$$
$$y_c=e^t(c_1\cos t+c_2\sin t)$$
$$y_p=\frac{e^t\sin t}{D^2-2D+2}\ldots\ldots\ldots\ldots D\to D+1$$
$$=e^t\frac{\sin t}{D^2+1}=e^t\frac{\sin t}{-1^2+1}\quad (Dr=0)$$
$$=e^t t\frac{\sin t}{2D}=\frac{-e^t t\cos t}{2}$$
$$GS\ is\ \ y=e^t(c_1\cos t+c_2\sin t)+\frac{-e^t t\cos t}{2}$$
$$y=x(c_1\cos(\log x)+c_2\sin(\log x))-\frac{x\log x\cos\log x}{2}$$

3) Solve $\frac{dy}{dx}-\frac{dx}{dy}=\frac{x}{y}-\frac{y}{x}$ **(Jan 2016, June 2015)**

Sol: The given equation can be written as

$$we\ note\ that\ p=\frac{dy}{dx}$$

$$p-\frac{1}{p}=\frac{x}{y}-\frac{y}{x}$$

$$p^2-p\left(\frac{x}{y}-\frac{y}{x}\right)-1=0$$

$$Solving\ \ for\ \ p\ we\ get$$

$$p=\frac{x}{y}\ \ or\ \ p=-\frac{y}{x}$$

$$now\ \frac{dy}{dx}=\frac{x}{y}\Rightarrow\int xdx-\int ydy=k$$

$$then\ \left(x^2-y^2-c\right)=0$$

$$Similarly\ \ \frac{dy}{dx}=-\frac{y}{x}\Rightarrow\int\frac{dx}{x}+\int\frac{dy}{y}=k$$

$$then\ \log x+\log y=\log c$$

$$then\ \left(xy-c\right)=0$$

$$\therefore GS\ is\ \left(x^2-y^2-c\right)\left(xy-c\right)=0$$

4) Solve $\left(x+a\right)^2y''-4\left(x+a\right)y'+6y=x$ **(July 2015)**

Solution: $put\ x=e^t\Rightarrow t=\log x$

$$put\ x+a=e^t\Rightarrow t=\log\left(x+a\right)$$

$$then\ we\ have\ \left[D(D-1)-4D+6\right]y=e^t-a$$

$$AE\quad m^2-5m+6=0\Rightarrow m=2,3$$

$$y_c=\ c_1e^{2t}+c_2e^{3t}$$

$$y_p=\frac{e^t-a}{D^2-5D+6}$$

$$=\frac{e^t}{2}-\frac{a}{6}$$

$$GS \ \ is \ \ y = c_1 e^{2t} + c_2 e^{3t} + \frac{e^t}{2} - \frac{a}{6}$$

$$y = c_1 (x+a)^2 + c_2 (x+a)^3 + \frac{x+a}{2} - \frac{a}{6}$$

5) Solve $p = \tan\left(x - \frac{p}{1+p^2}\right)$ **(Jan 2016)**

Sol: The given equation can be written as

$$x = \tan^{-1} p + \frac{p}{1+p^2} ....(1)$$

Differentiating both the sides

$$\frac{dx}{dy} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{(1+p^2)\frac{dp}{dy} - p\left(2p\frac{dp}{dy}\right)}{(1+p^2)^2}$$

$$\frac{1}{p} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{\frac{dp}{dy} - p^2\frac{dp}{dy}}{(1+p^2)^2}$$

$$= \frac{dp}{dy}\left[\frac{1}{1+p^2} + \frac{1-p^2}{(1+p^2)^2}\right]$$

$$= \frac{dp}{dy}\left[\frac{2}{(1+p^2)^2}\right]$$

$$dy = \frac{2p}{(1+p^2)^2}dp ....put \ 1+p^2 = t$$

$$\int dy = \int dt/t^2$$

$$y = \frac{-1}{t} + c \Rightarrow y = \frac{-1}{1+p^2} + c......(2)$$

(1) *and* (2) *are the general solution*

6) Find the general and singular solution of the equation $y = px + p^3$ **(Jan 2016)**

Solution: The given equation is Clairaut's equation

$\therefore$ the general solution is obtained by replacing p by c

$y = cx + c^3$

diff w.r.t c to get the singular solution

$0 = x + 3c^2 \Rightarrow c = \pm\sqrt{\frac{x}{3}}i$

$\therefore y = \sqrt{\frac{x}{3}}ix - \frac{x}{3}\sqrt{\frac{x}{3}}i$

7) *Solve* $(px - y)(py + x) = a^2 p$ *by reducing to Clairaut's equation.* **(June 2015)**

Solution: Put $u=x^2 \Rightarrow du = 2xdx \Rightarrow dx = \frac{du}{2x}$

Put $v=y^2 \Rightarrow dv = 2ydy \Rightarrow dy = \frac{dv}{2y}$

$$\therefore p = \frac{dy}{dx} = \frac{dv/2y}{du/2x} = \frac{x}{y}\frac{dv}{du} = \frac{x}{y}P \quad \text{where } P = \frac{dv}{du}$$

substituting this in the given equation we get

$$\left[\frac{x}{y}P\,x - y\right]\left[\frac{x}{y}P\,y + x\right] = a^2\frac{x}{y}P$$

$$\left[\frac{x^2P - y^2}{y}\right]\;P\,x + x\; = a^2\frac{x}{y}P$$

$$x(x^2P - y^2)(P+1) = Pa^2x \Rightarrow\; uP - v\;\;(P+1) = Pa^2$$

$$v = uP - \frac{Pa^2}{P+1} \text{......this is Clairaut's form}$$

The General Solution of this equation is

$$v = uc - \frac{ca^2}{c+1} \;\; or \;\; y^2 = cx^2 - \frac{ca^2}{c+1}$$

8) *Solve* $(1+x)^2 y'' + (1+x)y' + y = 2\sin \; \log x$ **(June 2015)**

Solution: *Put* $1+x = e^z \Rightarrow z = \log(1+x)$

$\therefore$ *given equation becomes*

$D(D-1)y + Dy + y = 2\sin z \Rightarrow \; D^2 + 1 \; y = 2\sin z$

The AE is $m^2 + 1 = 0 \Rightarrow m = \pm i$

$$\therefore y_c = c_1 \cos z + c_2 \sin z = c_1 \cos\log(1+x) + c_2 \sin\log(1+x)$$

$$y_p = \frac{2\sin z}{D^2+1} = 2z\frac{\sin z}{2D} = -z\cos z = -\log(1+x)\cos\log(1+x)$$

$$\therefore y = c_1 \cos\log(1+x) + c_2 \sin\log(1+x) - \log(1+x)\cos\log(1+x)$$

9) *Solve* $y = 2px + y^2 p^3$ *by solving for x.* **(June 2015)**

Solution: $x = \dfrac{y - y^2 p^3}{2p}$

differentiating w.r.t y we get

$$\frac{dx}{dy} = \frac{1}{2}\left[\frac{p\left(1 - y^2.3p^2\,\frac{dp}{dy} - p^3.2y\right) - \; y - y^2 p^3 \;\; \frac{dp}{dy}}{p^2}\right]$$

$$or \;\; \frac{1}{p} = \frac{1}{2p^2}\left[p - 3y^2p^3\frac{dp}{dy} - 2p^4 y - y\frac{dp}{dy} + y^2p^3\frac{dp}{dy}\right]$$

on cross multiplication

$$2p = p - 2y^2p^3\frac{dp}{dy}2p^4 y - y\frac{dp}{dy}$$

$$or \;\; p \;\; 1 + 2yp^3 \;\; + y\frac{dp}{dy} \;\; 1 + 2yp^3 \;\; = 0$$

$$\Rightarrow \left(p + y\frac{dp}{dy}\right) \; 1 + 2yp^3 \;\; = 0 \quad or \quad \frac{d}{dy}(py) = 0$$

$$on\,\text{int}\,egration \; p = \frac{c}{y}$$

substituting in given equation we get

$$y = 2.\frac{c}{y}x + y^2.\frac{c^3}{y^3} \Rightarrow y = 2.\frac{c}{y}x + \frac{c^3}{y}$$

or $y^2 = 2cx + c^3$ *which is the required solution*

10) *Solve* $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$ **(June 2015)**

Solution: *Put* $x = e^z \Rightarrow z = \log x$

$\therefore$ *given equation becomes*

$D(D-1)y + 4Dy + 2y = 2\sin z \Rightarrow \; D^2 + 3D + 2 \;\; y = e^{e^z}$

The AE is $m^2 + 3m + 2 = 0 \Rightarrow m = -1, -2$

$\therefore y_c = c_1e^{-z} + c_2e^{-2z} = \frac{c_1}{x} + \frac{c_2}{x^2}$

$$y_p = \frac{e^{e^z}}{D+1 \quad D+2} = \frac{1}{D+2}\frac{1}{D+1}e^{e^z} = \frac{1}{D+2}\left[e^{-z}\int e^{e^z}e^z dz\right]$$

$$= \frac{1}{D+2}\left[e^{-z}e^{e^z}\right] = e^{-2z}\int \; e^{e^z}e^{-z} \;\; e^{2z}dz = e^{-2z}e^{e^z} = \frac{e^x}{x^2}$$

$$\therefore y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{e^x}{x^2}$$

11) *Solve* $\frac{dx}{dt} - 7x + y = 0, \quad \frac{dy}{dt} - 2x - 5y = 0$ **(June 2015)**

Solution: *Put* $D = \frac{d}{dt}$ *the given equations becomes*

$Dx - 7x + y = 0 \; ; \;\; Dy - 2x - 5y = 0$

or $(D-7)x + y = 0 \; ; \; -2x + (D-5)y = 0$

solving for y we get $D^2 - 12D + 37 \;\; y = 0$

The AE is $m^2 - 12m + 37 = 0 \Rightarrow m = 6 \pm i$

$\therefore\ y = e^{6t}\ \ c_1 \cos t + c_2 \sin t$

$substituting\ y\ in\ \dfrac{dy}{dt} - 2x - 5y = 0$

$x = \dfrac{1}{2}\left[\dfrac{dy}{dt} - 5y\right] = \dfrac{1}{2}\left[e^{6t}\ \ -c_1 \sin t + c_2 \cos t\ \ + 6e^{6t}\ \ c_1 \cos t + c_2 \sin t\ \ - 5e^{6t}\ \ c_1 \cos t + c_2 \sin t\ \ \right]$

$= \dfrac{e^{6t}}{2}\left[\ c_1 + c_2\ \ \cos t + \ \ c_2 - c_1\ \ \sin t\right]$

12) Solve $p^2 - 4x^5 p - 12x^4 y = 0$, obtain the singular solution also.. **(Jan 2015)**

Sol: The given equation is solvable for y only.

$p^2 - 4x^5 p - 12x^4 y = 0$ ....(1)

$y = \dfrac{p^2 + 4x^5 p}{12x^4} = f(x, p)$

$Differentiating\ (1)\ w.r.t.x,$

$2p\dfrac{dp}{dx} + 4x^5\dfrac{dp}{dx} + 20x^4 p - 12x^4 p - 48x^3 y = 0$

$\dfrac{dp}{dx}(2p + 4x^5) + 8x^3(xp - \dfrac{p^2 + 4x^5 p}{2x^4}) = 0$

$(p + 2x^5)\dfrac{dp}{dx} = \dfrac{2p}{x}(p + 2x^5)$

$\dfrac{dp}{dx} - \dfrac{2p}{x} = 0$

$\Rightarrow Integrating\quad \log\sqrt{p} - \log x = k$

$\Rightarrow p = c^2 x^2 \quad \therefore\ equation\ (1)\ becomes$

$c^4 + 4c^2 x^3 = 12y$

$Setting\ c^2 = k\ the\ general\ solution\ becomes$

$k^2 + 4kx^3 = 12y$

$Differentiating\ w.r.t\ k\ partially\ we\ get$

$2k+4x^3=0$

Using $k=-2x^3$ in general solution we get

$x^6+3y=0$ as the singular solution

13) Solve $(px-y)(py+x)=2p$, by reducing into Clairaut's form,taking the substitution $X=x^2, Y=y^2$. **(Jan 2015)**

Solution: Let $X=x^2 \Rightarrow \frac{dX}{dx}=2x$

$$Y=x^2 \Rightarrow \frac{dX}{dy}=2y$$

$$Now,\ p=\frac{dy}{dx}=\frac{dy}{dY}\ \frac{dY}{dX}\ \frac{dX}{dx}\ and\ let\ P=\frac{dY}{dx}$$

$$P=\frac{1}{2y}.p.2x\ or\ \ p=\frac{x}{y}P$$

$$p=\frac{\sqrt{X}}{\sqrt{Y}}P$$

$$Consider\ (px-y)(py+x)=2p$$

$$\left[\frac{\sqrt{X}}{\sqrt{Y}}P\sqrt{X}-\sqrt{Y}\right]\left[\frac{\sqrt{X}}{\sqrt{Y}}P\sqrt{Y}+\sqrt{X}\right]=2\frac{\sqrt{X}}{\sqrt{Y}}P$$

$$(PX-Y)(P+1)=2P$$

$$Y=PX-\frac{2P}{P+1}$$

Is in the Clairaut's form and hence the associated general solution is

$$Y=cX-\frac{2c}{c+1}$$

Thus the required general solution of the given equation is $y^2=cx^2-\frac{2c}{c+1}$

14) Solve $p^3-4xyp+8y^2=0$ by solving for x.. **(Jan 2015)**

Solution: The given equation is solvable for x only.

$$p^3-4xyp+8y^2=0$$

$$x=\frac{p^3+8y^2}{4yp}=f(y,p)$$

$$Differentiating\ (1)\ w.r.t.y,$$

$$3p^2\frac{dp}{dy}-4xy\frac{dp}{dy}-4yp.\frac{1}{p}-4px+16y=0$$

$$\frac{dp}{dy}(3p^2 - 4xy) = 4px - 12y$$

$$\frac{dp}{dy}\left[3p^2 \frac{p^3 + 8y^2}{p}\right] = \left[\frac{p^3 + 8y^2}{y} - 12y\right]$$

$$\frac{dp}{dy}\left[\frac{2p^3 - 8y^2}{p}\right] = \frac{p^3 - 4y^2}{y}$$

$$\frac{2}{p}\frac{dp}{dy}(p^3 - 4y^2) = \frac{p^3 - 4y^2}{y}$$

$$\frac{2}{p}\frac{dp}{dy} = \frac{1}{y}$$

$$2\log p = \log y + \log c$$

$$Using\ P = \sqrt{cy}\ in (1)\ we\ have,$$

$$cy\sqrt{cy} - 4xy\sqrt{cy} + 8y^2 = 0$$

$$Dividing throughout\ by\ y\sqrt{y} = y^{3/2}\ we\ have,$$

$$c\sqrt{c} - 4x\sqrt{c} + 8\sqrt{y} = 0$$

$$\sqrt{c}(c - 4x) = -8\sqrt{y}$$

$$Thus\ the\ general\ solution\ is\ c(c - 4x)^2 = 64y$$

15) Solve $p(p+y) = x(x+y)$ **(June 2014)**

Sol: The given equation is, $p^2 + py - x(x+y) = 0$

$$p = \frac{-y \pm \sqrt{y^2 + 4x(x+y)}}{2}$$

$$p = \frac{-y \pm \sqrt{4x^2 + 4xy + y^2}}{2} = \frac{-y \pm (2x+y)}{2}$$

$$ie.,\ p = x\ \ or\ p = \frac{-2(y+x)}{2} = -(y+x)$$

$We\ have,$

$$\frac{dy}{dx} = x \Rightarrow y = \frac{x^2}{2} + k$$

$$Also,\ \frac{dy}{dx} = -y + x$$

$$ie., \ \frac{dy}{dx} + y = -x, is\, a\, linear\, d.e\, (similar\, to\, the\, previous\, problem)$$

$$P = 1,\ Q = -x; e^{\int P\,dx} = e^x$$

$$Hence\ \ ye^x = \int -xe^x dx + c$$

$$ie.,\ ye^x = -(xe^x - e^x) + c,\ integrating\ \ by\ \ parts.$$

$$Thus\, the\, general\, solution\, is\, given\, by\, (2y - x^2 - c)\left[e^x(y + x - 1) - c\right] = 0$$

16) Obtain the general solution and singular solution of the equation $y = 2px + p^2 y$. **(June 2014)**

Solution: The given equation is solvable for x and it can be written as

$$2x = \frac{y}{p} - py........(1)$$

Differentiating w.r.t y we get

$$\frac{2}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - p - y\frac{dp}{dy}$$

$$\Rightarrow \left(\frac{1}{p} + p\right)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$$

$$Ignoring \left(\frac{1}{p} + p\right) which\, does\, not\, contain\, \frac{dp}{dy}, this\, gives$$

$$1 + \frac{y}{p}\frac{dp}{dy} = 0 \quad or \quad \frac{dy}{y} + \frac{dp}{p} = 0$$

Integrating we get

$$yp = c........(2)$$

$$substituting\ \ for\ \ p\ \ from\ \ 2\ \ in\ \ (1)$$

$$y^2 = 2cx + c^2$$

17) Obtain the general solution and singular solution of the Clairaut's equation $xp^3 - yp^2 + 1 = 0$. **(Dec 2013)**

Solution: The given equation can be written as

$$y = \frac{xp^3 + 1}{p^2} \Rightarrow y = px + \frac{1}{p^2}\, is\, in\, the\, Clairaut's\ form\, y = px + f(p)$$

$$whose\, general\, solution\, is\, y = cx + f(c)$$

$$Thus\, general\, solution\, is\, y = cx + \frac{1}{c^2}$$

$$Differnetiating\ \ partially\, w.r.t.c\, we\, get$$

$$0 = x - \frac{2}{c^3} \Rightarrow c = \left(\frac{2}{x}\right)^{1/3}$$

Thus general solution becomes

$$y = \left(\frac{2}{x}\right)^{1/3} x + \left(\frac{x}{2}\right)^{2/3} \Rightarrow 2^{2/3} y = 3x^{2/3}$$

$$or \ \ 4y^3 = 27x^2$$

18) Solve $p^2 + 2py\cot x = y^2$ **.** **(Dec 2013)**

Solution: Dividing throughout by p^2, the equation can be written as

$$\frac{y^2}{p^2} - \frac{2y}{p}\cot x = 1 \ \ adding \cot^2 x \, to \, b.s$$

$$\frac{y^2}{p^2} - \frac{2y}{p}\cot x + \cot^2 x = 1 + \cot^2 x$$

$$or \quad \left(\frac{y}{p} - \cot x\right)^2 = \operatorname{cosec}^2 x$$

$$\Rightarrow \frac{y}{p} - \cot x = \pm \operatorname{cosec} x$$

$$\Rightarrow \frac{y}{dy/dx} = \cot x \pm \operatorname{cosec} x$$

$$\Rightarrow \frac{dy}{y} = \frac{\sin x}{\cos x + 1} dx \ \ and \ \ \frac{dy}{y} = \frac{\sin x}{\cos x - 1}$$

Integrating these two equations we get

$$y(\cos x + 1) = c_1 \ \ and \ \ \ y(\cos x - 1) = c_2$$

general solution is

$$y(\cos x + 1) - c \quad y(\cos x - 1) - c \ \ = 0$$

MODULE 3

PARTIAL DIFFERENTIAL EQUATION

1) **Form the partial differential equation of Z = y f(x) + xg(y) where f and g are arbitrary functions.** **(Jan 2016)**

$$Sol: \frac{\partial z}{\partial x} = g(y) + yf'(x);$$

$$\frac{\partial z}{\partial y} = xg'(y) + f(x)$$

Substituting $g'(y)$ *and* $f'(x)$

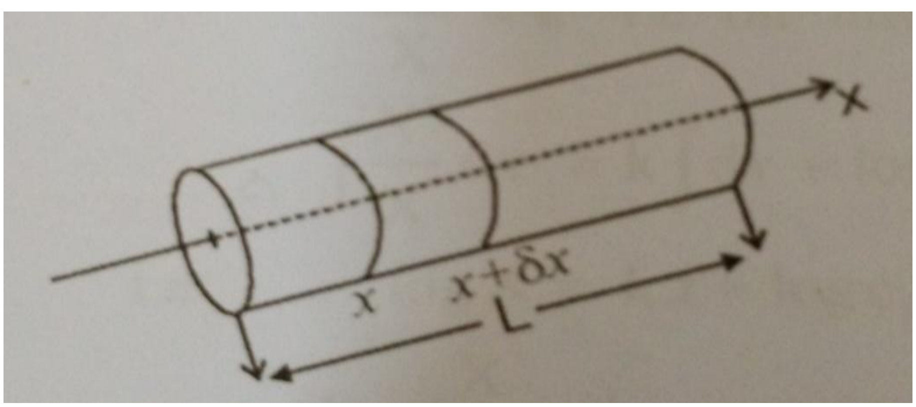
$$xy\frac{\partial^2 z}{\partial x \partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - [xg(y) + yf(x)]$$

$$xy\frac{\partial^2 z}{\partial x \partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z$$

This is the required pde.

2. Derive one dimensional heat equation as $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ **.** **(Jan 2016, July 2015)**

Sol:



Consider a heat conducting homogeneous rod of length L placed along x-axis. One end of the rod at x=0(Origin) and the other end of the rod at x=L.

Assume that the rod as constant density ρ and uniform cross section A. Also assume that the rod is insulated laterally and therefore heat flows only in the x direction. The rod is sufficiently thin so that the temperature is same at all points of any cross sectional area of the rod.

Let u(x, t) be the temperature of the cross section at the point x at any time t.

The amount of heat crossing any section of the rod per second depends on the area A of the cross section, the thermal conductivity k of the material of rod and the temperature gradient $\frac{\partial u}{\partial x}$

i.e., the rate of change of temperature with respect to distance normal to the area.

Therefore q_1 the quantity of heat flowing into the cross section at a distance x in unit time is

$$q_1 = -kA\left(\frac{\partial u}{\partial x}\right)_x \text{per second}$$

Negative sign appears because heat flows in the direction of decreasing temperature (as x increases u decreases)

q_2 the quantity of heat flowing out of the cross section at a distance $x+\delta x$
(i.e, the rate of heat flow at cross section $x+\delta x$)

$$q_2 = -kA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \text{per second}$$

The rate of change of heat content in segment of the rod between x and x+□ x must be equal to net heat flow into this segment of the rod is

$$q_1 - q_2 = kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] \text{per second}............(1)$$

But the rate of increase of heat in the rod

$$s\rho A\delta x\frac{\partial u}{\partial t}...............(2)$$

Where S is the specific heat, ρ the density of material.

$\therefore$ From (1) & (2)

$$s\rho A\delta x\frac{\partial u}{\partial t}=kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial u}{\partial x}\right)_{x}\right]$$

$$or\ s\rho\frac{\partial u}{\partial t}=k\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial u}{\partial x}\right)_{x}}{\delta x}\right]$$

Taking limit as $\delta x\to 0$, we have

$$s\rho\frac{\partial}{\partial t}=k\frac{\partial^2 u}{\partial x^2}\ or\ \frac{\partial u}{\partial t}=\frac{k}{s\rho}\frac{\partial^2 u}{\partial x^2}$$

$$or\ \frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}\text{...........................(3)}\quad \text{where } c^2=\frac{k}{s\rho}$$

Is known as diffusivity constant.

Equation (3) is the one dimensional heat equation which is second order homogenous and parabolic type.

3. From the function f(x^2+ y^2 , z-xy) = 0 form the partial differential equation . (July 2015)

Sol: Let u = x^2+ y^2 and v= z-xy so that the given relation is f(u,v) = 0

Differentiating this partially w.r.t x and y, we get

$$\frac{\partial f}{\partial u}(2x)+\frac{\partial f}{\partial v}\left(\frac{\partial z}{\partial x}-y\right)=0$$

$$\frac{\partial f}{\partial u}(2y)+\frac{\partial f}{\partial v}\left(\frac{\partial z}{\partial y}-x\right)=0$$

Eliminating $\frac{\partial f}{\partial u}$ *and* $\frac{\partial f}{\partial v}$ *from these equations, we get*

$$\begin{vmatrix} 2x & \frac{\partial z}{\partial x}-y \\ 2y & \frac{\partial z}{\partial y}-x \end{vmatrix}=0$$

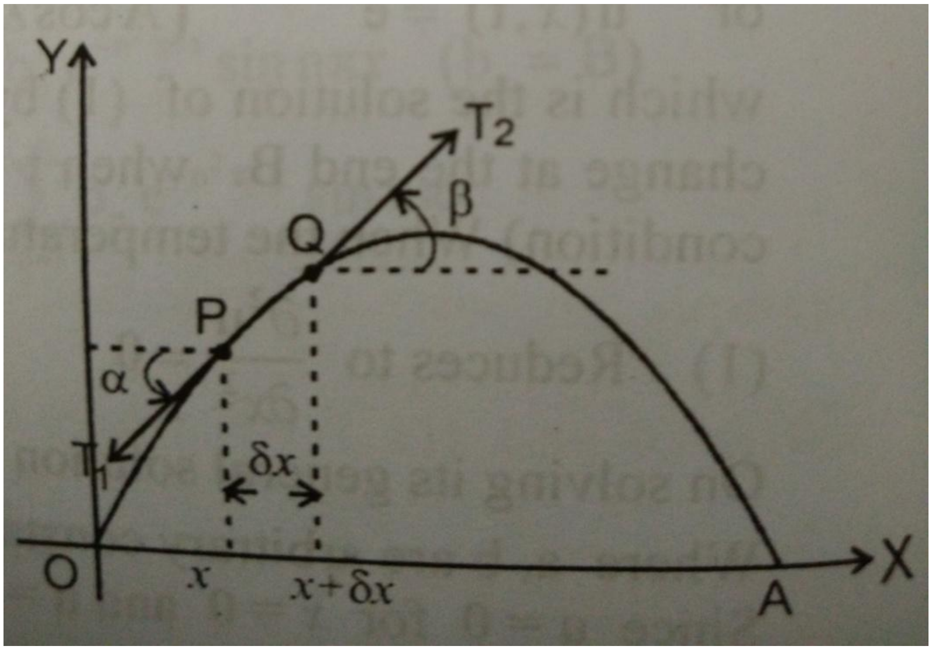
$$or\ x\left(\frac{\partial z}{\partial y}-x\right)-y\left(\frac{\partial z}{\partial x}-y\right)=0$$

$$or\ x\left(\frac{\partial z}{\partial y}\right)-y\left(\frac{\partial z}{\partial x}\right)=x^2-y^2$$

This is the required partial differential equation.

4) Derive one dimensional wave equation as $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ **.** **(July 2015)**

Sol:



Consider a tightly stretched elastic string of length l stretched between two points O and A and displaced slightly from its equilibrium position OA. Taking O as origin and OA as x axis and a perpendicular line through O as Y- axis. We shall find the displacement y a function of the distance x and the time t.

We shall obtain the equation of motion of string under the following assumptions.

The string is perfectly flexible and offers no resistance to bending

Points on the string move only in the vertical direction, there is no motion in the horizontal direction. The motion takes place entirely in the X Y plane .

Gravitational forces on the string are neglected.

Let m be the mass per unit length of the string. Consider the motion of an element PQ of length δs . Since the string does not offer resistance to bending, the tensions T_1

At P and T_2 at Q are tangential to the curve.

Since the is no motion in the horizontal direction, some of the forces in the horizontal direction must be zero.

i.e., $-T_1\cos\alpha + T_2\cos\beta = 0$ or $T_1\cos\alpha = T_2\cos\beta = T = \text{constant}$…..(1)

Since gravitational force on the string is neglected , the only two forces acting on the string are the vertical components of tension - $T_1\sin$□ at P and $T_2\sin\beta$ at Q with up[ward direction takes as positive.

Mass of an element PQ is m δs . By Newton's second law of motion , the equation of motion in the vertical direction is

Resultant of forces = mass *acceleration

$T_2\sin\beta$ - $T_1\sin$□ $=m\delta s\dfrac{\partial^2 y}{\partial t^2}$..............(2) .

$$\frac{2}{1}\, gives\, \frac{T_2\sin\beta}{T_2\cos\beta}-\frac{T_1\sin\alpha}{T_1\cos\alpha}=\frac{m\delta s}{T}\frac{\partial^2 y}{\partial t^2}$$

$$or\, \tan\beta-\tan\alpha=\frac{m\delta s}{T}\frac{\partial^2 y}{\partial t^2}$$

.

$$\frac{\partial^2 y}{\partial t^2}=\frac{T}{m\delta s}\ \tan\beta-\tan\alpha$$

$$\frac{\partial^2 y}{\partial t^2}=\frac{T}{m\delta s}\left[\left(\frac{\partial y}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial y}{\partial x}\right)_{x}\right]$$

$(\because \delta s=\delta x\, to\, a\ first\, approximation\, and\, \tan\alpha\, ,\tan\beta\, are\, the\, slopes\, of\ the\, curve\, of\ the\, string\, at\, x\, and\ x+\delta x)$

$$\frac{\partial^2 y}{\partial t^2}=\frac{T}{m}\left[\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial y}{\partial x}\right)_{x}}{\delta x}\right]$$

$Taking\ Limit\ as\, \delta x\to 0$

$$\frac{\partial^2 y}{\partial t^2}=\frac{T}{m}\frac{\partial^2 y}{\partial x^2}\qquad or\qquad \frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}\text{...................(3)}\quad \text{where } c^2=\frac{T}{m}$$

Which is the partial differential equation giving the transverse vibrations of the string .

Equation (3) is the one dimensional wave equation which is second order homogenous and parabolic type.

5) Solve $z_{xy}=\sin x\sin y\ for\ which\, z_y=-2\sin y\, when\, x=0\, and\ z=0$

when y is an odd multiple of $\dfrac{\pi}{2}$. **(Jan 2015)**

Solution: Here we first find z by integration and apply the given conditions to determine the arbitrary functions occurring as constants of integration.

The given PDF can be written as $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right)=\sin x\sin y$

Integrating w.r.t x treating y as constant,

$$\frac{\partial z}{\partial y} = \sin y \int \sin x\, dx + f(y) = -\sin y \cos x + f(y)$$

Integrating w.r.t y treating x as constant

$$z = -\cos x \int \sin y\, dy + \int f(y)\, dy + g(x)$$

$$z = -\cos x\ (-\cos y) + F(y) + g(x),$$

$$where\, F(y) = \int f(y)\, dy.$$

$$Thus\ z = \cos x \cos y + F(y) + g(x)$$

$$Also\, by\ data, \frac{\partial z}{\partial y} = -2\sin y\ \ when\, x = 0.\ U\sin g\ this\, in\, (1)$$

$$-2\sin y = (-\sin y).1 + f(y)\ (\cos 0 = 1)$$

$$Hence\, F(y) = \int f(y)\, dy = \int -\sin y\, dy = \cos y$$

$$With\, this, (2)\, becomes\, z = \cos x \cos y + \cos y + g(x)$$

$$U\sin g\ \ the\, condition\, that\, z = 0\ \ if\ \ y = (2n+1)\frac{\pi}{2}\, in\, (3)\, we\ \ have$$

$$0 = \cos x \cos(2n+1)\frac{\pi}{2} + \cos x\, c(2n+1)\frac{\pi}{2} + g(x)$$

$$But\, \cos(2n+1)\frac{\pi}{2} = 0. and\ \ hence\, 0 = 0 + 0 + g(x)$$

Thus the solution of the PDE is given by

z=cos x cosy + cosy

6) Solve: $x^2 - y^2 - z^2\ \ P + 2xyq = 2xz$ **(Jan 2015)**

Solution: The given equation is of the form Pp +Qq = R.
The auxiliary equations are

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

Taking the second and third terms we have,

$$\frac{dy}{2xy} = \frac{dz}{2xz}\ \ or\ \ \frac{dy}{y} = \frac{dz}{z}$$

Integrating we get, log y = log z + log C_1

Log (y/z) = logc_1

Using multipliers x, y, z each ratio in (1) is equal to

$$\frac{xdx+y\,dy+z\,dz}{x^3-xy^2-xz^2+2xy^2+2xz^2}=\frac{x\,dx+y\,dy+z\,dz}{x^3+xy^2+xz^2}=\frac{x\,dx+y\,dy+z\,dz}{x\ \ x^2+y^2+z^2}$$

Let us consider

$$\frac{dy}{2\,xy}=\frac{xdx+y\,dy+z\,dz}{x(x^2+y^2+z^2)}$$

$$Integrating\ we\ get,\ \log y=\log(x^2+y^2+z^2)+\log c_2$$

Thus a general solution of the PDE is given by

$$\phi(y/z,\ y/x^2+y^2+z^2)=0$$

7) Solve by the method of variables $3u_x+2u_y=0, given\,that\,u(x,0)=4e^{-x}$ **(Jan 2015)**

Solution: *Given* $3\frac{\partial u}{\partial x}+2\frac{\partial u}{\partial x}=0$

Assume solution of (1) as

U=XY where X=X(x); $Y=Y(y)$

$$3\frac{\partial u}{\partial x}(xy)+2\frac{\partial u}{\partial x}(xy)=0$$

$$\Rightarrow 3Y\frac{dX}{dx}+2X\frac{dY}{dy}=0\Rightarrow\frac{3}{X}\frac{dX}{dx}=\frac{-2}{Y}\frac{dY}{dy}$$

$$Let\ \frac{3}{X}\frac{dX}{dx}=K\ \Rightarrow\frac{3dX}{X}=kdx$$

$$\Rightarrow 3\log X=kx+c_1\Rightarrow\log X=\frac{Kx}{3}+c_1$$

$$\Rightarrow X=e^{\frac{kx}{3}+c_1}$$

$$Let\ \frac{-2}{Y}\frac{dY}{dy}=k\quad\Rightarrow\frac{dY}{Y}=\frac{-Kdy}{2}$$

$$\Rightarrow\log Y=\frac{-Kdy}{2}+c_2\Rightarrow Y=e^{\frac{-ky}{2}+c_2}$$

Substituting (2) & (3) *in* (1)

$$U=e^{K\left(\frac{x}{3}-\frac{y}{2}\right)+c_1+c_2}$$

$$Also\ u(x_1o)=4e^{-x}$$

$$i.e.,\ 4e^{-x} = Ae^{k\left(\frac{2x}{6}\right)} \Rightarrow 4e^{-x} = Ae^{\frac{kx}{3}}$$

$$Comparing\ we\ get\ A = 4\ \&\ K = -3$$

$$U = 4e^{-3\left(\frac{x}{3}-\frac{y}{2}\right)}$$ is required solution.

8) Form the partial differential equation by eliminating the arbitrary functions from z = f(y-2x)+g(2y-x) (June 2014)

Solution: By data, **z = f(y-2x)+g(2y-x)**

$$p = \frac{\partial z}{\partial x} = -2f'(y-2x) - g'(2y-x)$$

$$q = \frac{\partial z}{\partial y} = f'(y-2x) + 2g'(2y-x)$$

$$r = \frac{\partial^2 z}{\partial x^2} = 4f''(y-2x) + g''(2y-x)..............(1)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = -2f''(y-2x) - 2g''(2y-x).........(2)$$

$$t = \frac{\partial^2 z}{\partial y^2} = f''(y-2x) + 4g''(2y-x)................(3)$$

$$(1)\times 2 + (2) \Rightarrow 2r + s = 6f''(y-2x)..............(4)$$

$$(2)\times 2 + (3) \Rightarrow 2s + t = -3f''(y-2x)............(5)$$

$$Nowdividing\,(4)\ by\ (5)\,we\,get$$

$$\frac{2r+s}{2s+t} = -2 \quad or\,2r + 5s + 2t = 0$$

$$Thus \quad 2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0\,is\,the\,required\ PDE$$

9) Solve: $x^2 - yz \;\; P + \;\; y^2 - zx \;\; q = z^2 - xy$ **(June 2014, Dec 2013)**

Solution: The given equation is of the form Pp +Qq = R.

The auxiliary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$$

Equivalently we can write in the form,

$$\frac{dx - dy}{x^2 - y^2 \;\; + z \;\; x - y} = \frac{dy - dz}{y^2 - z^2 \;\; + x \;\; y - z} = \frac{dz - dx}{z^2 - x^2 \;\; + y \;\; z - x}$$

$$i.e., \frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z)} = \frac{dz-dx}{(z-x)(x+y+z)}$$

$$or\ \frac{dx-dy}{x-y} = \frac{dy-dz}{y-z} = \frac{dz-dx}{z-x} \text{..........(1)}$$

From the first and second terms of (1) we have,

$$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z} \Rightarrow \log\left(\frac{x-y}{y-z}\right) = \log c_1 \Rightarrow \frac{x-y}{y-z} = c_1$$

$$Similarly\ we\ get\ \frac{y-z}{z-x} = c_2$$

Thus the general solution is

$$\phi\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0$$

11) Solve by the method of variables $4\frac{\partial u}{dx} + \frac{\partial u}{\partial y} = 3u$, *given that* $u(0, y) = 2e^{5y}$

(Jan 2015, June2014)

Solution: *Given* $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$

Assume solution of (1) as

$u = XY$ *where* $X = X(x); Y = Y(y)$

$$4\frac{\partial}{\partial x}(XY) + \frac{\partial}{\partial y}(XY) = 3XY$$

$$\Rightarrow 4Y\frac{dX}{dx} + X\frac{dY}{dy} = 3XY \Rightarrow \frac{4}{X}\frac{dX}{dx} + \frac{1}{Y}\frac{dY}{dy} = 3$$

$$Let\ \frac{4}{X}\frac{dX}{dx} = k,\quad 3 - \frac{1}{Y}\frac{dY}{dy} = k$$

Separating variables and integrating we get

$$\Rightarrow \log X = \frac{kx}{4} + c_1,\quad \log Y = (3-k)y + c_2$$

$$\Rightarrow X = e^{\frac{kx}{4}+c_1}\ and\ Y = e^{(3-k)y+c_2}$$

$$Hence\ u = XY = e^{c_1+c_2}e^{\frac{kx}{4}+(3-k)y} = Ae^{\frac{kx}{4}+(3-k)y}\quad where\ A = e^{c_1+c_2}$$

put $x = 0$ *and* $u = 2e^{5y}$

The general solution becomes

$$2e^{5y} = Ae^{(3-k)y} \Rightarrow A = 2\ and\ k = -2$$

$\therefore$ *Particular solution is*

$$u = 2e^{\frac{-x}{2}+5y}$$

11) **12) Solve** $\frac{\partial^2 z}{\partial x^2} + z = 0 \; given\, that \; when \; x = 0, z = e^y \; and \; \frac{\partial z}{\partial x} = 1$ **(Dec 2013)**

Solution: Let us suppose that z is a function of x only, the PDE assumes the form of ODE

$$\frac{d^2 z}{dx^2} + z = 0 \Rightarrow \; D^2 + 1 \; z = 0 \; where\, D = \frac{d}{dx}$$

$$A.E. \;\; m^2 + 1 = 0 \Rightarrow m = \pm i$$

The solution of the ODE

$$z = c_1 \cos x + c_2 \sin x$$

$$z = f(y)\cos x + g(y)\sin x....(1) by\; replacing\; c_1 \,\&\, c_2 \; by \; functions\, of \; y$$

$$Now\; put\; x = 0 \; and\; z = e^y \; in\, (1)$$

$$\therefore e^y = f(y)\cos 0 + g(y)\sin 0 \Rightarrow e^y = f(y)$$

$$Again \;\; put\; x = 0 \; and \; \frac{\partial z}{\partial x} = 1$$

$$Differentiating\, (1) \; partially\; w.r.t \; x$$

$$\frac{\partial z}{\partial x} = -f(y)\sin x + g(y)\cos x \Rightarrow 1 = -f(y)\sin 0 + g(y)\cos 0$$

$$\therefore g(y) = 1$$

equation (1) becomes

$$z = e^y \cos x + \sin x$$

MODULE-4
INTEGRAL CALCULUS

1. Evaluate $\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$ **changing the order of integration.** **(July 2015)**

$$Sol: Let\ I = \int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$$

To identify the region of integration R let us find the point of intersection of the curves

$$y = x^2 \ \ and \ \ y = 2 - x$$

$$\Rightarrow x = 1, x = -2 \ \ and \ \ y = 1, y = 4$$

On changing the order we must have constant limits for y and variable limits for to cover the same region

$$I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} x\,y\,dx\,dy + \int_{y=1}^{2} \int_{x=0}^{2-y} x\,y\,dx\,dy$$

$$= \int_{y=0}^{1} y\left[\frac{x^2}{2}\right]_{x=0}^{\sqrt{y}} dy + \int_{y=1}^{2} y\left[\frac{x^2}{2}\right]_{x=0}^{2-y} dy$$

$$= \int_{y=0}^{1} \frac{y^2}{2} dy + \frac{1}{2}\int_{y=1}^{2} y\,(2-\mathrm{y})^2\,dy$$

$$= \left[\frac{y^3}{6}\right]_0^1 + \frac{1}{2}\left[2y^2 - \frac{4y^3}{3} + \frac{y^4}{4}\right]_1^2$$

$$\therefore I = \frac{3}{8}$$

2. *Evaluate* $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a} x^2 + y^2 + z^2 \ dxdydz$ (Jan 2016)

$$Solution: I = \int_{-c}^{c}\int_{-b}^{b}\left[x^2 z + y^2 z + \frac{z^3}{3}\right]_{-a}^{a} dydx$$

$$= \int_{-c}^{c}\int_{-b}^{b}\left[2ax^2 + 2ay^2 + \frac{2a^3}{3}\right] dydx$$

$$= \int_{-c}^{c}\left[2ax^2 y + 2a \ \ y^3/3 \ + \frac{2a^3}{3}y\right]_{-b}^{b} dx$$

$$= \int_{-c}^{c}\left[4abx^2 + 4a\ \ b^3/3\ \ + \frac{4a^3b}{3}\right]dx$$

$$= \left[4ab\ \ x^3/3\ \ + 4a\ \ b^3/3\ \ x + \frac{4a^3b}{3}x\right]_{-c}^{c}$$

$$= \frac{8abc^3}{3} + \frac{8ab^3c}{3} + \frac{8a^3bc}{3}$$

$$= \frac{8abc}{3}\ \ a^2 + b^2 + c^2$$

3) $Solve\ \frac{\partial^2 z}{\partial x\partial y} = Sinx\sin y,\ for\ which\ \frac{\partial z}{\partial y} = -2\sin y\ \ when\ x = 0\ and\ z = 0$

$when\ y\ is\ an\ odd\ multiple\ of\ \frac{\pi}{2}$ **Jan 2016, July2015**

$Sol: \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = Sinx\sin y,$

Integrating $w.r.t\ x\ treating\ y\ as$ constant

$\frac{\partial z}{\partial y} = \sin y\int \sin x dx + f(y)$

$\frac{\partial z}{\partial y} = -\sin y\cos x + f(y)$.........(1)

integrating $w.r.t\ y\ treating\ x\ as$ constant

$z = -\cos x(-\cos y) + F(y) + g(x)\ \ where F(y) = \int f(y)dy$

$\therefore z = \cos x\cos y + F(y) + g(x)$........(2)

by given data (1) becomes

-2siny = -siny.1 + f(y) $\rightarrow$ f(y) = -siny

hence F(y) = $\int$-sinydy = cosy

(2) $\Rightarrow z = \cos x\cos y + \cos y + g(x)$........(3)

again by data $0 = \cos x\cos(2n+1)\frac{\pi}{2} + \cos(2n+1)\frac{\pi}{2} + g(x)$

$but\ \cos(2n+1)\frac{\pi}{2} = 0,\ hence\ g(x) = 0$

$solution\ of\ \ given\ PDE\ is$

$z = \cos y(\cos x + 1)$

4. Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z} x+y+z \ \ dy\,dx\,dz$ **.** **(July 2015)**

$$Sol: Let\ I=\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z} x+y+z \ \ dy\,dx\,dz$$

$$I=\int_{-1}^{1}\int_{0}^{z}\left(xy+\frac{y^2}{2}+zy\right)_{x-z}^{x=z} dx\,dz$$

$$=\int_{-1}^{1}\int_{0}^{z} 4xz+2z^2 \ \ dx\,dz$$

$$=\int_{-1}^{1}\left[z(2x^2)+2z^2(\mathrm{x})\right]_0^z dz$$

$$=\int_{-1}^{1}4z^3\,dz$$

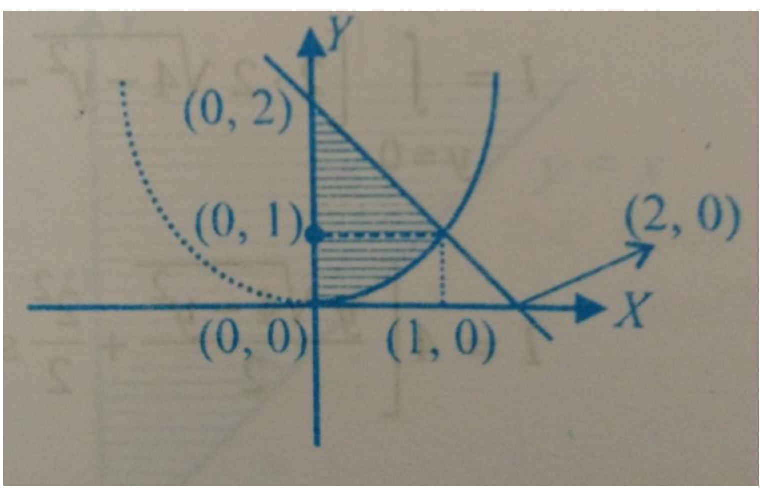
$$=\left[z^4\right]_{-1}^{1}$$

$$=0$$

5. Evaluate $\iint_R xy\,dx\,dy$ **, where R is the region bounded by x- axis, the ordinate x=2a and the** parabola $x^2=4ay$. **(Jan 2016)**

Sol: $x^2=4ay$ is a parabola symmetrical about the y-axis. The point of intersection of this curve with x=2a is to be found. Hence $4a^2=4ay$ or y=a

The point of intersection is (2a,a)



$$I = \iint_R xy\, dx\, dy$$

$$= \int_{x=0}^{2a} \int_{y=0}^{\frac{x^2}{4a}} xy\, dy\, dx$$

$$= \int_{x=0}^{2a} x \left[\frac{y^2}{2} \right]_0^{\frac{x^2}{4a}} dx$$

$$= \frac{1}{2} \int_{x=0}^{2a} \left[\frac{x^5}{16a^2} \right] dx$$

$$= \frac{1}{32a^2} \left[\frac{x^6}{6} \right]_0^{2a}$$

$$\therefore\ I = \frac{a^4}{3}$$

6. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$ **by changing into polar coordinates. (Jan 2016)**

Solution. In polars we have $x = r\cos\theta$, $y = r\sin\theta$

$\therefore$ $x^2 + y^2 = r^2$ and $dx\, dy = r\, dr\, d\theta$

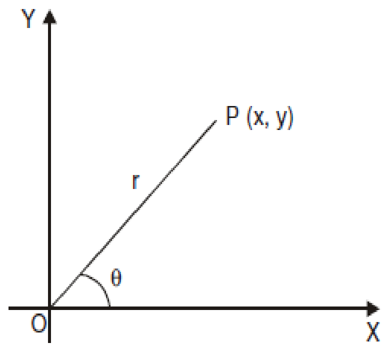
Since x, y varies from 0 to ∞

r also varies from 0 to ∞

In the first quadrant 'θ'

varies from 0 to $\pi/2$

Thus $I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} r\, dr\, d\theta$



Put $r^2 = t \quad \therefore\ r\,dr = \frac{dt}{2}$

t also varies from 0 to ∞

$$I = \int_{\theta=0}^{\pi/2} \int_{t=0}^{\infty} e^{-t}\, \frac{dt}{2}\, d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi/2} \left[-e^{-t}\right]_0^{\infty} d\theta$$

$$= \frac{-1}{2} \int_0^{\pi/2} (0-1)\, d\theta$$

$$= +\frac{1}{2} \int_0^{\pi/2} 1 \,.\, d\theta$$

$$= \frac{+1}{2} [\theta]_0^{\pi/2} = \frac{+1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4} \cdot$$

7) Find the area between the parabolas y^2=4ax and x^2 = 4ay **(July 2015)**

Sol: We have y^2=4ax ………………… (1) and x2 = 4ay…………………(2).

Solving (1) and (2) we get the point of intersections (0,0) and (4a,4a) .

The shaded portion in the figure is the required area divide the arc into horizontal strips of width ∂y

x varies from p, $\frac{y2}{4a}$ to $Q\sqrt{4ay}$ and then y varies from O, y=0 to A, y=4a .

Therefore the required area is

$$\int_0^{4a} dy \int_{\frac{y^2}{4a}}^{\sqrt{4ay}} dx = \int_0^{4a} dy \ \ x \Big|_{\frac{y^2}{4a}}^{\sqrt{4ay}}$$

$$= \int_0^{4a}\left[\sqrt{4ay} - \frac{y^2}{4a}\right]dy = \left[\sqrt{4a}.\frac{y^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1}{4a}\frac{y^3}{3}\right]_0^{4a}$$

$$= \left[\frac{4\sqrt{a}}{3}(4a)^{\frac{3}{2}} - \frac{1}{12a}(4a)^3\right]$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2$$

8) *Define Gamma function and Beta function.* Prove *that* $\Gamma(1/2) = \sqrt{\pi}$ (Jan 2016)

Sol : $\Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1}dx,\ (n > 0)$ *is called Gamma function.*

$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx,\ (m,n > 0)$ *is called Beta function.*

We have $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, *put* $m = n = 1/2$

$$\beta(1/2,1/2) = \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1)}\ but\ \Gamma(1) = 1$$

$$\therefore \beta(1/2,1/2) = \left\{\Gamma(1/2)\right\}^2$$

now consider $\beta(m,n) = 2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$

$$\beta(1/2,1/2) = 2\int_0^{\pi/2}\sin^0\theta\cos^0\theta d\theta = 2\int_0^{\pi/2}1d\theta = \pi$$

$$\therefore \pi = \left\{\Gamma(1/2)\right\}^2 \Rightarrow \Gamma(1/2) = \sqrt{\pi}$$

9) Prove that $\beta(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ (Jan 2016)

Solution: We have by the definition of Beta and the Gamma function

$$\beta(m,n)=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta.........(1)$$

$$\Gamma(n)=2\int_0^{\infty}e^{-x^2}x^{2n-1}dx..............(2)$$

$$\Gamma(m)=2\int_0^{\infty}e^{-y^2}y^{2m-1}dx............(3)$$

$$\Gamma(m+n)=2\int_0^{\infty}e^{-r^2}r^{2(m+n)-1}dx........(4)$$

$$\Gamma(m).\Gamma(n)=4\int_0^{\infty}\int_0^{\infty}e^{-(x^2+y^2)}x^{2n-1}y^{2m-1}dxdy.........(5)$$

evaluating RHS by changing into *polars put* $x=r\cos\theta, y=r\sin\theta$

$$x^2+y^2=r^2,\ dxdy=rdrd\theta,\ r:0\to\infty \text{ and } \theta:0\to\pi/2$$

$$(5)\Rightarrow\Gamma(m).\Gamma(n)=4\int_0^{\infty}\int_0^{\pi/2}e^{-r^2}(r\cos\theta)^{2n-1}(r\sin\theta)^{2m-1}rdrd\theta$$

$$=4\int_0^{\infty}\int_0^{\pi/2}e^{-r^2}(r^{2m+2n-1}\cos^{2n-1}\theta)(\sin\theta)^{2m-1}drd\theta$$

$$=\left[2\int_0^{\infty}e^{-r^2}r^{2(m+n)-1}dr\right]\left[2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta\right]$$

$$=\Gamma(m+n).\beta(m,n)\ \text{by using (1) and (4)}$$

$$Thus\ \beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

10) Show that the area between the parabolas $y^2=4ax$, and $x^2=4ay$ is $\dfrac{16}{3}a^2$ (July 2015, Jan 2016)

Sol: Solving $y^2=4ax$ and $x^2=4ay$ we get

$$y:0\to 4a \text{ and } x:y^2/4a\to\sqrt{4ay}$$

$$\therefore \text{required area}=\int_0^{4a}dy\int_{y^2/4a}^{\sqrt{4ay}}dx=\int_0^{4a}dy\ \left[x\right]_{y^2/4a}^{\sqrt{4ay}}$$

$$=\int_0^{4a}\left(\sqrt{4ay}-y^2/4a\right)dy$$

$$= \left[\sqrt{4a}\,\frac{y^{3/2}}{3} - \frac{1}{4a}\frac{y^3}{3} \right]_0^{4a}$$

$$= \left[\frac{4\sqrt{a}}{3} \left(4a\right)^{3/2} - \frac{1}{12a}\left(4a\right)^{3} \right]$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2$$

$$= \frac{16}{3}a^2$$

11) Find the volume common to the cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$ **(Jan 2016)**

Sol: In the given region z varies from $-\sqrt{a^2-x^2}$ to $+\sqrt{a^2-x^2}$ and y varies from $-\sqrt{a^2-x^2}$ to $+\sqrt{a^2-x^2}$ .for z=0, y=0 x varies from –a to a

Therefore, required volume is

$$v = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dzdydx$$

$$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left(z \right)_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dydx$$

$$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2-x^2}\,dydx$$

$$= 2\int_{-a}^{a} \left\{ \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\,dy \right\} dx$$

$$= 2\int_{-a}^{a} \left[\sqrt{a^2-x^2}\,y \right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$

$$= 2\int_{-a}^{a} \left(\sqrt{a^2-x^2}\; 2\sqrt{a^2-x^2} \right) dx$$

$$= 4\int_{-a}^{a} a^2 - x^2 \quad dx = 4\left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a}$$

$$= 4\left[\left(a^3 - \frac{a^3}{3}\right) - \left(-a^3 + \frac{a^3}{3}\right)\right]$$

$$= 4\left[2a^3 - \frac{2a^3}{3}\right] = \frac{16a^3}{4}$$

12) *Evaluate* $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$ using *beta and gamma function* **(June 2015)**

Solution: *Put* $x^2 = \sin\theta \Rightarrow x = \sqrt{\sin\theta} \Rightarrow dx = \frac{1}{2\sqrt{\sin\theta}}\cos\theta d\theta$

$$I = \int_0^{\pi/2} \frac{\cos\theta}{2\sqrt{\sin\theta}\sqrt{1-\sin^2\theta}} d\theta = \int_0^{\pi/2} \frac{1}{2}\sin^{-\frac{1}{2}}\theta d\theta$$

$$= \frac{1}{2}\frac{\sqrt{\pi}}{2}\frac{\Gamma\frac{1}{4}}{\Gamma\frac{3}{4}}$$

MODULE-5
LAPLACE TRANSFORMS

1. Find $L\left\{e^{-2t}\sin 3t + e^{t}t\cos t\right\}$ **.** **(Jan 2016)**

$$Sol.\ L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\{e^{-2t}\sin 3t\} = \frac{3}{s^2+4s+12}$$

$$L\{\cos t\} = \frac{s}{s^2+1}$$

$$L\{t\cos t\} = (-1)\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$$

$$= -\left\{\frac{(s^2+1) - s(2s)}{(s^2+1)^2}\right\}$$

$$= -\left\{\frac{1-s^2}{s^2+1}\right\} = \frac{s^2-1}{s^2+1}$$

$$L\{e^{-t}t\cos t\} = \frac{(s+1)^2-1}{(s+1)^2+1}$$

$$= \frac{s^2+2s}{s^2+2s+2}$$

$$\therefore L\{e^{-2t}\sin 3t + e^{-t}t\cos t\} = \frac{3}{s^2+4s+12} + \frac{s^2+2s}{s^2+2s+2}$$

2. Find the inverse Laplace transform of $\frac{4s+5}{(s-1)^2(s+2)}$. **(Jan 2016)**

$$Sol.\ L^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\}$$

$$\frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2}$$

$$4s+5=A(s+1)(s+2)+B(s+2)+C(s+1)^2$$

$$on\ \ solving\ ,\ \ A=3, B=1, C=-3$$

$$L^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\}=3L^{-1}\left\{\frac{1}{s+1}\right\}+L^{-1}\left\{\frac{1}{(s+1)^2}\right\}-3L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$=3e^{-t}+e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\}-3e^{-2t}$$

$$=3e^{-t}+e^{-t}.t-3e^{-2t}$$

3. Solve $y''+6y'+9y=12t^2e^{-3t}$ by Laplace transform method with y(0) = 0 =y^1(0). (Jan 2016)

$$Sol.\ y''(t)+6y'(t)+9y(t)=12t^2e^{-3t}$$

$$L[y''(t)]+6L[y'(t)]+9L[y(t)]=12L\left[e^{-3t}t^2\right]$$

$$s^2L[y(t)]-sy(0)-y'(0)+6[sL[y(t)]-y(0)]+9L[y(t)]=\frac{12\cdot 2!}{(s+3)^3}$$

using given initial conditions we obtain,

$$(s^2+6s+9)L[y(t)]=\frac{24}{(s+3)^3}$$

$$L[y(t)]=\frac{24}{(s+3)^5}$$

$$y(t)=L^{-1}\left(\frac{24}{(s+3)^5}\right)$$

$$y(t)=24e^{-3t}L^{-1}\left(\frac{1}{s^5}\right)=24e^{-3t}\frac{t^4}{4!}$$

$$\therefore y(t)=e^{-3t}t^4$$

4. Express $f(t)=\begin{cases}\cos t, 0<t\le\pi\\ 1\quad ,\pi<t\le 2\pi\\ \sin t, t>2\pi\end{cases}$ **in terms of unit step function and hence find its Laplace transform . (Jan 2016)**

$$sol:\ f(t)=\cos t+(1-\cos t)u(t-\pi)+(\sin t-1)u(t-2\pi)$$

$$L\{f(t)\}=L\{\cos\}t+L\{(1-\cos t)u(t-\pi)\}+L\{(\sin t-1)u(t-2\pi)\}.....(1)$$

$$Let\,F(t-\pi)=1-\cos t\qquad ;\ G(t-2\pi)=\sin t-1$$

$$F(t)=1-\cos(t+\pi)\qquad ;\ G(t)=\sin(t+2\pi)-1$$

$$F(t)=1-\cos t\qquad ;\ G(t)=\sin t-1$$

$$\therefore \overline{F}(s)=\frac{1}{s}+\frac{s}{s^2+1}\quad ;\ \overline{G}(s)=\frac{1}{s^2+1}-\frac{1}{s}$$

$$L[F(t-\pi)u(t-\pi)]=e^{-\pi s}\overline{F}(s)$$

$$L[G(t-2\pi)u(t-2\pi)]=e^{-2\pi s}\overline{G}(s)$$

$$L[(1-\cos t)\,u(t-\pi)]=e^{-\pi s}\left(\frac{1}{s}+\frac{s}{s^2+1}\right)$$

$$L[(\text{sint}-1)u(t-2\pi)]=e^{-2\pi s}\left(\frac{1}{s^2+1}-\frac{1}{s}\right)$$

$$\therefore equ(1)\ \text{becomes}$$

$$L\{f(t)\}=\frac{s}{s^2+1}+e^{-\pi s}\left(\frac{1}{s}+\frac{s}{s^2+1}\right)+e^{-2\pi s}\left(\frac{1}{s^2+1}-\frac{1}{s}\right)$$

5. Solve $y''+6y'+9y=12t^2e^{-3t}$ by Laplace transform method with y(0) = 0 =y^1(0). (Jan 2016)

$$Sol.\,y''(\text{t})+6y'(\text{t})+9y(\text{t})=12t^2e^{-3t}$$

$$L\ \ y''(\text{t})\ +6L\ \ y'(\text{t})\ +9L\ \ y(\text{t})\ =12L\left[e^{-3t}t^2\right]$$

$$s^2L\ \ y(\text{t})\ -sy(0)-y'(0)\ +6\ \ sL\ \ y(\text{t})\ -y(0)\ +9L\ \ y(\text{t})\ =\frac{12\cdot 2!}{s+3\ ^{3}}$$

using given initial conditions we obtain,

$$s^2+6s+9\ \ L\ \ y(\text{t})\ =\frac{24}{s+3\ ^{3}}$$

$$L\ \ y(\text{t})\ =\frac{24}{s+3\ ^{5}}$$

$$y(\text{t})=L^{-1}\left(\frac{24}{s+3\ ^{5}}\right)$$

$$y(\text{t})=24\,\text{e}^{-3t}\,L^{-1}\left(\frac{1}{s\ ^{5}}\right)=24\,\text{e}^{-3t}\,\frac{t^4}{4!}$$

$$\therefore y(\text{t})=\text{e}^{-3t}\,t^4$$

6. Find $L\left\{\dfrac{\cos at-\cos bt}{t}\right\}$ **(Jan 2016)**

$Sol: Let\ f(t)=\text{cosat}-\text{cosbt}$

$$\therefore \bar{f}(s)=\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}$$

$$hence\ L\left[\frac{f(t)}{t}\right]=\int_s^{\infty}\left(\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right)ds$$

$$L\left[\frac{\text{cosat}-\text{cosbt}}{t}\right]=\frac{1}{2}\left[\log(s^2+a^2)-\log(s^2+b^2)\right]_s^{\infty}$$

$$=\frac{1}{2}\left[\frac{\log(s^2+a^2)}{\log(s^2+b^2)}\right]_s^{\infty}$$

$$=\frac{1}{2}\left[\lim_{s\to\infty}\log\left(\frac{1+a^2/s^2}{1+b^2/s^2}\right)-\log\left(\frac{(s^2+a^2)}{(s^2+b^2)}\right)\right]$$

$$=\frac{1}{2}\left[\log 1-\log\left(\frac{(s^2+a^2)}{(s^2+b^2)}\right)\right]$$

$$=\frac{1}{2}\left[\log\left(\frac{(s^2+b^2)}{(s^2+a^2)}\right)\right]$$

$$L\left[\frac{\text{cosat}-\text{cosbt}}{t}\right]=\log\sqrt{\frac{(s^2+b^2)}{(s^2+a^2)}}.$$

7. Find the Laplace transform of $te^{-4t}\sin 3t\ and\ \dfrac{e^{at}-e^{-at}}{t}$ **(July 2015)**

$$Sol:(i)L(\sin 3t)=\frac{3}{s^2+9}\therefore L(te^{-4t}\sin 3t)=\frac{3}{(s+4)^2+9}=\frac{3}{s^2+8s+25}$$

$$Hence\ L(te^{-4t}\sin 3t)=\frac{-d}{ds}\left\{\frac{3}{s^2+8s+25}\right\}=\frac{3(2s+8)}{(s^2+8s+25)^2}$$

$$thus\ L(te^{-4t}\sin 3t)=\frac{6(s+4)}{(s^2+8s+25)^2}$$

$(ii)\ Let\ f(t)=e^{at}-e^{-at}$

$$\therefore L\{f(t)\}=L(e^{at})-L(e^{-at})=\frac{1}{s-a}-\frac{1}{s+a}=F(s)$$

$$L\left\{\frac{e^{at}-e^{-at}}{t}\right\} = \int_s^\infty\left(\frac{1}{s-a}-\frac{1}{s+a}\right)ds = \left[\log(s-a)-\log(s+a)\right]_s^\infty$$

$$= \lim_{s\to\infty}\log\left(\frac{s-a}{s+a}\right) - \log\left(\frac{s-a}{s+a}\right)$$

$$= \lim_{s\to\infty}\log\left(\frac{1-\frac{a}{s}}{1+\frac{a}{s}}\right) - \log\left(\frac{s-a}{s+a}\right)$$

$$= \log\left(\frac{s+a}{s-a}\right)$$

8. Express f(t) in terms of unit step function and find its Laplace transform given that

$$f(t) = t\begin{cases} t^2, & 0<t<2 \\ 4t, & 2<t<4 \\ 8, & t>4 \end{cases}$$

(July 2015)

$$Sol: f(t) = f_1(t) + [f_2(t) - f_1(t)]\,u(t-a) + [f_3(t) - f_2(t)]\,u(t-b)$$

$$= t^2 + \left[4t-t^2\right]u(t-2) + (8-4t)\,u(t-4)$$

$$L\{f(t)\} = L\{t^2\} + L\left\{\left[4t-t^2\right]u(t-2)\right\} + L\{(8-4t)\,u(t-4)\} \quad .....(1)$$

$$\text{Let } F_1(t-2) = 4t-t^2$$

$$F_1(t) = 4(t+2) - (t+2)^2$$

$$= 4t+8-t^2-4t-4$$

$$= -t^2+4$$

$$L\{F_1(t)\} = \frac{4}{s}-\frac{2}{s^3}$$

$$L\left\{\left[4t-t^2\right]u(t-2)\right\} = e^{-2s}\left(\frac{4}{s}-\frac{2}{s^3}\right) \quad ....(2)$$

$$Let\ F_2(t-4) = 8-4t$$

$$F_2(t) = 8-4(t+4) = -4t-8$$

$$L\{F_2(t)\} = -\frac{4}{s^2}-\frac{8}{s}$$

$$L\{(8-4t)\,u(t-4)\} = e^{-4s}\left(-\frac{4}{s^2}-\frac{8}{s}\right)$$

$$= -4e^{-4s}\left(\frac{1}{s^2}+\frac{2}{s}\right) \quad ......(3)$$

substituting (3) and (2) in (1)

$$L\{f(t)\} = \frac{2!}{s^3} + e^{-2s}\left(\frac{4}{s}-\frac{2}{s^3}\right) - 4e^{-4s}\left(\frac{1}{s^2}+\frac{2}{s}\right)$$

9. Find $L^{-1}\left\{\frac{1}{(s+1)(s^2+9)}\right\}$ **using convolution theorem.** **(July 2015)**

$$Sol: Let\ F(s)=\frac{1}{s+1}\ and\ G(s)=\frac{1}{s^2+9}$$

$$f(t)=L^{-1}[F(s)]=L^{-1}\left\{\frac{1}{s+1}\right\}=e^{-t}$$

$$g(t)=L^{-1}[\mathrm{G}(s)]=L^{-1}\left\{\frac{1}{s^2+9}\right\}=\frac{1}{3}\sin 3t$$

$$\therefore L^{-1}\left\{\frac{1}{(s+1)(s^2+9)}\right\}=L^{-1}[F(s)\mathrm{G}(s)]=f(t)*g(t)$$

$$=\int_0^t f(t-u)g(u)\,\mathrm{du}=\int_0^t e^{-(t-u)}\frac{1}{3}.\sin 3u\,du$$

$$=\frac{1}{3}e^{-t}\int_0^t e^{u}\sin 3u\,du=\frac{1}{3}e^{-t}\left[\frac{\mathrm{e}^{u}(\sin 3u-3\cos 3u)}{5}\right]_0^t$$

$$=\frac{1}{15}e^{-t}\left\{e^{t}(\sin 3t-3\cos 3t)+3\right\}$$

$$=\frac{1}{15}\left\{\sin 3t-3\cos 3t+3e^{-t}\right\}$$

10. A periodic function f(t) with period 2 is defined by $f(t)=\begin{cases} t & ,0<t<1 \\ 2-t, & 1<t<2\end{cases}$ **find L{f(t)}**

(July 2015)

$$Sol: We\ have\ T=2\ and\ L[f(t)]=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$$

$$L[f(t)]=\frac{1}{1-e^{-sT}}\int_0^2 e^{-st}f(t)dt$$

$$=\frac{1}{1-e^{-2s}}\left\{\int_0^1 te^{-st}dt+\int_1^2 (2-t)e^{-st}dt\right\}$$

$$= \frac{1}{1-e^{-2s}}\left\{\left[t.\frac{e^{-st}}{-s}-(1)\frac{e^{-st}}{s^2}\right]_0^1+\left[\ 2-t\ .\frac{e^{-st}}{-s}-(-1)\frac{e^{-st}}{s^2}\right]_1^2\right\}$$

$$=\frac{1}{s^2\ 1-e^{-2s}}\ \ -2e^{-s}+1+e^{-2s}$$

$$=\frac{1-e^{-s}\ ^2}{s^2\ 1-e^{-s}\ \ 1+e^{-s}}$$

$$L\ \ f(t)\ =\frac{1-e^{-s}}{s^2\ \ 1+e^{-s}}=\frac{e^{as/2}-e^{-as/2}}{s^2\ \ e^{as/2}+e^{-as/2}}$$

$$\therefore\ L\ \ f(t)\ =\frac{2\sinh(as/2)}{s^2.2\cosh(as/2)}=\frac{1}{s^2}\tanh(as/2)$$

11. Find $L^{-1}\left\{\frac{5s-2}{3s^2+4s+8}+\log\left(\frac{1}{s^2}-1\right)\right\}$ **(July 2015)**

$$Sol: L^{-1}\left\{\frac{5s-2}{3s^2+4s+8}+\log\left(\frac{1}{s^2}-1\right)\right\}=L^{-1}\left\{\frac{5s-2}{3s^2+4s+8}\right\}+L^{-1}\left\{\log\left(\frac{1}{s^2}-1\right)\right\}...(1)$$

$$L^{-1}\left\{\frac{5s-2}{3s^2+4s+8}\right\}\quad ,$$

$$consider\ 3s^2+4s+8=3\left[s^2+\frac{4}{3}s+\frac{8}{3}\right]=3\left[\left(s+\frac{2}{3}\right)^2+\frac{20}{9}\right]$$

$$5s-2=5\left(s+\frac{2}{3}\right)-\frac{16}{3}$$

$$L^{-1}\left\{\frac{5s-2}{3s^2+4s+8}\right\}=\frac{1}{3}L^{-1}\left\{\frac{5\left(s+\frac{2}{3}\right)-\frac{16}{3}}{\left(s+\frac{2}{3}\right)^2+\frac{20}{9}}\right\}$$

$$=\frac{e^{-\frac{2t}{3}}}{3}\left\{5L^{-1}\left[\frac{s}{s^2+\left(\frac{\sqrt{20}}{3}\right)^2}\right]-\frac{16}{3}L^{-1}\left[\frac{1}{s^2+\left(\frac{\sqrt{20}}{3}\right)^2}\right]\right\}$$

$$=\frac{e^{-\frac{2t}{3}}}{3}\left\{5\cos\frac{\sqrt{20}}{3}t-\frac{16}{\sqrt{20}}\sin\frac{\sqrt{20}}{3}t\right\}.....(2)$$

$$\overline{f}(s) = \log\left(\frac{1-s^2}{s^2}\right) = \log(1-s^2) - 2\log s$$

$$-\overline{f}'(s) = 2\left\{\frac{1}{s} - \frac{s}{s^2-1}\right\}$$

$$L^{-1}\left[-\overline{f}'(s)\right] = 2L^{-1}\left\{\frac{1}{s} - \frac{s}{s^2-1}\right\}$$

$$\therefore\ f(t) = \frac{2(1-\cosh t)}{t} \text{........}(3)$$

$$(1) \Rightarrow L^{-1}\left\{\frac{5s-2}{3s^2+4s+8} + \log\left(\frac{1}{s^2}-1\right)\right\} = \frac{e^{-\frac{2t}{3}}}{3}\left\{5\cos\frac{\sqrt{20}}{3}t - \frac{16}{\sqrt{20}}\sin\frac{\sqrt{20}}{3}t\right\} + \frac{2(1-\cosh t)}{t}$$

12. Solve using Laplace transform method $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = te^{-t}$ *with* $y(0)=1, y'(0)=-2$

(July 2015)

$$Sol:\ y''(t) + 2y'(t) + y(t) = te^{-t}$$

$$L\{y''(t)\} + 2L\{y'(t)\} + L\{y(t)\} = L\left[e^{-t}t\right]$$

$$s^2L\{y(t)\} - sy(0) - y'(0) + 2\{sL\{y(t)\} - y(0)\} + L\{y(t)\} = \frac{1}{(s+1)^2}$$

using given initial conditions we obtain,

$$(s^2+2s+1)L\{y(t)\} - s + 2 - 2 = \frac{1}{(s+1)^2}$$

$$(s+1)^2 L\{y(t)\} = s + \frac{1}{(s+1)^2}$$

$$L\{y(t)\} = \frac{s}{(s+1)^2} + \frac{1}{(s+1)^4}$$

$$y(t) = \frac{s+1}{(s+1)^2} - \frac{1}{(s+1)^2} + \frac{1}{(s+1)^4}$$

$$= e^{-t}\left\{L^{-1}\left[\frac{1}{s}\right]-L^{-1}\left[\frac{1}{s^2}\right]+L^{-1}\left[\frac{1}{s^4}\right]\right\}$$

$$\therefore\ y(\mathrm{t}) = \mathrm{e}^{-t}\left(1-t+\frac{t^3}{6}\right)$$

13) *Find* $L\ t(\sin^3 t-\cos^3 t)$ (Jan 2015)

Solution: $$\sin^3 t-\cos^3 t=\frac{1}{4}(3\sin t-\sin 3t)-\frac{1}{4}(3\cos t-\cos 3t)$$

$$L(\sin^3 t-\cos^3 t)=\frac{1}{4}\left\{\frac{3}{s^2+1}-\frac{3}{s^2+9}\right\}-\frac{1}{4}\left\{\frac{3s}{s^2+1}-\frac{s}{s^2+9}\right\}$$

$$U\sin g\ the\ property: L\ tf(t)\ = -\frac{d}{ds}\left[\bar{f}(s)\right]\ we\ have,$$

$$L\ t(\sin^3 t-\cos^3 t)\ = -\frac{3}{4}\left[\frac{-2s}{(s^2+1)^2}+\frac{2s}{(s^2+9)^2}\right]+\frac{1}{4}\left[3.\frac{(s^2+1)-2s^2}{(s^2+1)^2}+\frac{(s^2+9)-2s^2}{(s^2+9)^2}\right]$$

Thus

$$L\ t(\sin^3 t-\cos^3 t)\ = \frac{3s}{2}\left[\frac{1}{(s^2+1)^2}-\frac{1}{(s^2+9)^2}\right]+\frac{1}{4}\left[3.\frac{(1-s^2)}{(s^2+1)^2}+\frac{9-s^2}{(s^2+9)^2}\right]$$

14) Express f(t) in terms of unit step function and hence find the Laplace transform

$$given\ that\ f(t)=\begin{cases} t^2 & 0<t<2 \\ 4t & 2<t<4 \\ 8 & t>4 \end{cases}$$ **(Jan 2015)**

Solution: $$f(t)=t^2+(4t-t^2)u(t-2)+(8-4t)+(t-4)$$

$$L\ f(t)\ = L\ t^2\ + L\ (4t-t^2)u(t-2)\ + L\ (8-4t)\ (t-4)$$

$$=\frac{2}{s^3}+e^{-2s}L\ 4(t+2)-(t+2)^2\ + e^{-4s}L\ 8-4\ (t+4)$$

$$=\frac{2}{s^3}+e^{-2s}L\left[-t^2+4\right]+e^{-4s}L\ -4t-8$$

$$=\frac{2}{s^3}+e^{-2s}\left[\frac{4}{s}-\frac{2}{s^3}\right]-e^{-4s}\left[\frac{4}{s^2}-\frac{8}{s}\right]$$

15) Find the value of $\int_0^\infty t^3 e^{-t} \sin t dt$ using Laplace transforms **(Jan 2015)**

Solution: We have

$$\int_0^\infty e^{-st} t^3 \sin t dt = L(t^3 \sin t) = (-1)^3 \frac{d^3}{ds^3}\left[L(\sin t)\right] = -\frac{d^3}{ds^3}\left[\frac{1}{s^2+1}\right]$$

$$= -\frac{d^2}{ds^2}\left[\frac{-2s}{(s^2+1)^2}\right] = \frac{d}{ds}\left[\frac{(s^2+1)^2.2-2s.2(s^2+1)}{(s^2+1)^4}\right]$$

$$= \frac{d}{ds}\left[\frac{2s^2-2s}{(s^2+1)^3}\right] = \left[\frac{(s^2+1)^3(4s-2)-(2s^2-2s).3(s^2+1)^2}{(s^2+1)^6}\right]$$

$$= \frac{4s^3-8s^2+8s-2}{(s^2+1)^4}$$

Putting s = 1 in this result, we get

$$\int_0^\infty e^{-t} t^3 \sin t dt = \frac{1}{8}$$

This is the result as required.

16) Find Laplace transform of a periodic function $f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t, & \pi < t < 2\pi \end{cases}$ (**Jan 2015)**

Solution: Here T = 2 π. Therefore

$$L[f(t)] = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t) dt$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\int_0^\pi t e^{-st} dt + \int_\pi^{2\pi} (\pi - t) e^{-st} dt\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left\{\left[t\left(\frac{e^{-st}}{-s}\right) - \frac{e^{-st}}{s^2}\right]_0^\pi + \left[(\pi - t)\left(\frac{e^{-st}}{-s}\right) - (-1)\frac{e^{-st}}{s^2}\right]_\pi^{2\pi}\right\}$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{-\pi e^{-\pi s}}{s} - \frac{e^{-\pi s}}{s^2} + \frac{1}{s^2} + \frac{\pi e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2} - \frac{e^{-\pi s}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{\pi}{s}\ e^{-2\pi s} - e^{-\pi s}\ \ - \frac{1-2e^{-\pi s}+e^{-2\pi s}}{s^2}\right]$$

17) Prove that L $\delta(t-a) = e^{-as}$ **(June 2014)**

Solution: We shall first find the Laplace transform of $\delta_\varepsilon(t-a)$

$$L\ \delta_\varepsilon(t-a) = \int_0^\infty e^{-st}\delta_\varepsilon(t-a)dt$$

$$= \int_0^a e^{-st}\delta_\varepsilon(t-a)dt + \int_a^{a+\varepsilon} e^{-st}\delta_\varepsilon(t-a)dt + \int_{a+\varepsilon}^\infty e^{-st}\delta_\varepsilon(t-a)dt$$

$$i.e., L\ \delta_\varepsilon(t-a) = \int_a^{a+\varepsilon} e^{-st}\frac{1}{\varepsilon}dt, by\ using\ definition$$

$$= \frac{1}{\varepsilon}\left[\frac{e^{-st}}{-s}\right]_a^{a+\varepsilon} = \frac{-1}{\varepsilon s}\ e^{-s(a+\varepsilon)} - e^{-as}$$

$$\therefore L\ \delta_\varepsilon(t-a) = e^{-as}\left\{\frac{1-e^{-\varepsilon s}}{\varepsilon s}\right\}$$

$$But\ L\ \delta(t-a) = L\left[\lim_{\varepsilon\to 0}\delta_\varepsilon(t-a)\right] = \lim_{\varepsilon\to 0} L\left[\delta_\varepsilon(t-a)\right]$$

$$= e^{-as}\lim_{\varepsilon\to 0}\left\{\frac{1-e^{-\varepsilon s}}{\varepsilon s}\right\} = e^{-as}\ (By\ applying\ L'Hospital's\ rule)$$

18) $If\ f(t) = \begin{cases} t, & 0\le t\le a \\ 2a-t, & a\le t\le 2a \end{cases}, where\ f(t+2a) = f(t), show\ that\ L\ \ f(t) = \frac{E}{s}\tanh\left(\frac{as}{2}\right)$

(June 2014)

Solution: Here T = 2 a. Therefore $L\ \ f(t) = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)dt$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a te^{-st}dt + \int_a^{2a}(2a-t)e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left\{\left[t\left(\frac{e^{-st}}{-s}\right)-\frac{e^{-st}}{s^2}\right]_0^a+\left[(2a-t)\left(\frac{e^{-st}}{-s}\right)-(-1)\frac{e^{-st}}{s^2}\right]_a^{2a}\right\}$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{-ae^{-as}}{s}-\frac{e^{-as}}{s^2}+\frac{1}{s^2}+\frac{e^{-2as}}{s^2}+\frac{ae^{-as}}{s}-\frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{s^2(1-e^{-2as})}(1-2e^{-as}+e^{-2as}) = \frac{(1-e^{-as})^2}{s^2(1-e^{-as})(1+e^{-as})}$$

$$= \frac{(1-e^{-as})}{s^2(1+e^{-as})} = \frac{(e^{as/2}-e^{-as/2})}{s^2(e^{as/2}+e^{-as/2})} \text{(multiplied Nr. \& Dr by } e^{as/2})$$

$$\therefore L\{f(t)\} = \frac{2\sinh(as/2)}{s^2.2\cosh(as/2)} = \frac{1}{s^2}\tanh(as/2)$$

19) *Express* $f(t)=\begin{cases}1, & 0<t<1\\ t, & 1<t\le 2\\ t^2, & t>2\end{cases}$ *in terms of unit step function and hence find its Laplace transform* **(June 2014)**

Solution: $f(t)=1+(t-1)u(t-1)+(t^2-t)u(t-2)$

$$L\{f(t)\}=L\{1\}+L\{(t-1)u(t-1)\}+L\{(t^2-t)u(t-2)\}$$

$$=\frac{1}{s}+e^{-s}L\{t\}+e^{-2s}L\{t^2+3t+2\}$$

$$=\frac{1}{s}+e^{-s}\frac{1}{s^2}+e^{-2s}\left\{\frac{2}{s^3}+\frac{3}{s^2}+\frac{2}{s}\right\}$$

20) *Find the inverse Laplace transform of* $\tan^{-1}(2/s^2)$ **(June 2014)**

Solution: *Let* $\bar{f}(s)=\tan^{-1}(2/s^2)$

$$\therefore\ \bar{f}'(s)=\frac{1}{\tan^{-1}(4/s^4)}.\frac{-4}{s^3}=\frac{-4s}{s^4+4}$$

$$\text{Hence } L^{-1}\left[-\bar{f}'(s)\right]=L^{-1}\left[\frac{4s}{s^4+4}\right]$$

$$tf(t)=L^{-1}\left[\frac{4s}{s^4+4}\right]$$

$$Now\, s^4+4=(s^2+2)^2-4s^2=(s^2+2+2s)\ (s^2+2-2s)$$

$$Also\, 4s=(s^2+2+2s)-(s^2+2-2s)$$

$$Hence\, \frac{4s}{s^4+4}=\frac{(s^2+2+2s)-(s^2+2-2s)}{(s^2+2+2s)-(s^2+2-2s)}$$

$$=\frac{1}{s^2+2-2s}-\frac{1}{s^2+2+2s}$$

$$L^{-1}\left[\frac{4s}{s^4+4}\right]=L^{-1}\left[\frac{1}{s^2-2s+2}\right]-L^{-1}\left[\frac{1}{s^2+2s+2}\right]$$

$U\sin g(1)\, in\, L.H.S\ we\ have,$

$$tf(t)=L^{-1}\left[\frac{1}{(s-1)^2+1}\right]-L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$tf(t)=e^t L^{-1}\left[\frac{1}{s^2+1}\right]-e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$tf(t)=e^t\sin t-e^{-t}\sin t=\sin t(e^t-e^{-t})$$

$$tf(t)=sont.2\sinh t$$

$$f(t)=\frac{2\sin t\sin \text{h}\, t}{t}$$

21) $Find\, L^{-1}\left[\frac{S}{(s-1)(s^2+4)}\right] using\ convolution\ theorem$ **(June 2014)**

Solution:

$$L^{-1}\left[\frac{S}{(s-1)(s^2+4)}\right]\text{using convolution theorem.}$$

$$\text{F(s) = }\frac{1}{\text{s-1}}\qquad c_1(s)=\frac{\text{s}}{(s^2+4)}\, then$$

$$f(t)=L^{-1}\ F(s)\ =e^{+t}\ and\ g(t)=L^{-1}\ G(s)\ =\cos 2t$$

$$L^{-1}\left[\frac{S}{(s-1)(s^2+4)}\right]=L^{-1}\ F(s)G(s)\ =f(t)*g(t)$$

$$=\int_0^t f(t-u)g(u)du=\int_0^t e^{t-u}\cos 2u\ du$$

$$=e^t\int_0^t e^{-u}\cos 2u\ du$$

$$= e^t \left[\frac{e^{-u}}{1^2+2^2} \;(-1)\cos 2u + 2\sin 2u \right]_0^t$$

$$= \frac{e^t}{5}\left[e^{-t}(2\sin 2t - \cos 2t) + 1 \right]$$

$$= \frac{1}{5}(2\sin 2t - \cos 2t + e^{-t})$$

22) Solve the following initial value problem by using Laplace transforms:

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-t};\; y(0)=0, y'(0)=0$$

(Dec 2013)

Solution: The given equation is $y''(t) + 4y'(t) + 4y(t) = e^{-t}$

Taking Laplace transform on both sides we have,

$$L\;\; y''(t) \;\; + 4L\;\; y'(t)\;\; + 4L\;\; y(t)\;\; = L\;\; e^{-1}$$

$$s^2 L\;\; y(t)\;\; - sy(0) - y'(0)\;\; + 4\;\; sL\;\; y(t)\;\; - y(0))\;\; + 4L\;\; y(t)\;\; = \frac{1}{s+1}$$

using the given initial conditions we obtain,

$$L\;\; y(t)\;\;\; s^2 + 4s + 4\;\; = \frac{1}{s+1}$$

$$y(t) = L^{-1}\left[\frac{1}{(s+1)(s+2)^2} \right]$$

$$Let \frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+2\;\;^2}$$

Multiplying with $(s+1)(s+2)^2$ *we obtain*

$$1 = A(s+2)^2 + B(s+1)(s+2) + C(s+1)$$

Putting $s = -1$ *we get* $A = 1$

Putting $s = -2$ *we get* $C = -1$

Putting $s = 0$ *we have* $1 = 1(4) + B(2) - 1(1) \therefore B = -1$

$$Hence \frac{1}{(s+1)(s+2)^2} = \frac{1}{s+1} + \frac{-1}{s+2} + \frac{-1}{s+2\;\;^2}$$

$$L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right]=L^{-1}\left[\frac{1}{s+1}\right]-L^{-1}\left[\frac{1}{s+2}\right]-L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$y(t)=e^{-t}-e^{-2t}-e^{-2t}\,L^{-1}\left(\frac{1}{s^2}\right)$$

$$Thus\ \ y(t)=e^{-t}-e^{-2t}-e^{-2t}t=e^{-t}-(1+t)e^{-2t}$$

23) Find L $L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\}$

(Dec 2013)

Solution: $\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\}=\frac{A}{s-1}+\frac{Bs+C}{(s^2+2s+5)}$

$$5s+3=A(s^2+2s+5)+(Bs+C)(s-1)\Rightarrow put\ \ s=1\ and\ \ s=0\ \ we\ \ get\ \ A=1,C=2,B=-1$$

$$\frac{5s+3}{(s-1)(s^2+2s+5)}=\frac{1}{s-1}+\frac{3-(s+1)}{(s+1)^2+4}$$

$$Thus\ \ L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\}=L^{-1}\left[\frac{1}{s-1}\right]+L^{-1}\left[\frac{3-(s-1)}{(s+1)^2+4}\right]$$

$$=e^{t}+e^{-t}L^{-1}\left[\frac{3-s}{s^2+4}\right]=e^{t}+e^{-t}\left\{3L^{-1}\left[\frac{1}{s^2+4}\right]-L^{-1}\left[\frac{s}{s^2+4}\right]\right\}$$

$$L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\}=e^{t}+e^{-t}\left[\frac{3}{2}.\sin 2t-\cos 2t\right]$$

24) $U\sin g\ \ convolution\ \ theorem\ \ evaluate L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+a^2)}\right\}$

(Dec 2013)

$$Solution:\ \ f(t)=L^{-1}\left[\ddot{f}(s)\right]=\cos at; g(t)=L^{-1}\ \ \bar{g}(s)\ \ =\cos bt$$

Now applying convolution theorem we have,

$$L^{-1}\left\{\frac{s^2}{s^2+a^2\quad s^2+a^2}\right\}=\int_{u=0}^{t}\cos au.\cos(bt-bu)du$$

$$f(t)*g(t)=\frac{1}{2}\int_{u=0}^{t}\left[\cos(au+bt-bu)+\int_{u=0}^{t}\cos(au-bt+bu)du\right]$$

$$=\frac{1}{2}\left[\frac{\sin(au+bt-bu)}{a-b}+\frac{\sin(au-bt+bu)}{a+b}\right]_{u=0}^{t}$$

$$=\frac{1}{2}\left[\frac{1}{a-b}\ \sin at-\sin bt\ +\frac{1}{a+b}\ \sin at+\sin bt\ \right]$$

$$=\frac{1}{2}\left[\sin at.\frac{2a}{a^2-b^2}+\sin bt.\frac{-2b}{a^2-b^2}\right]$$

$$Thus\ L\left[f(t)*g(t)=\frac{1}{a^2+b^2}\left[a.\frac{a}{s^2+a^2}-b.\frac{b}{s^2+b^2}\right]\right]=\frac{s^2}{(s^2+a^2)(s^2+a^2)}=\bar{f}(s).\bar{g}(s)$$

$$Thus\quad L^{-1}\left\{\frac{s^2}{s^2+a^2\quad s^2+b^2}\right\}=\frac{a\sin at-b\sin bt}{a^2-b^2}$$

25) *Solve* $y'''+2y''-y'-2y=0$ *given* $y(0)=y'(0)=0$ *and* $y''(0)=6$ *by using Laplace transform method*

(Dec 2013)

Solution: *Taking laplace on both sides* $L\left[[y'''(t)]+L[2y''(t)]-L[y'(t)]-2L[y(t)]=l(0)\right]$

$$s^3 l[y(t)-s^2y(0)-sy'(0)-y''(0)\}+2\{s^2Ly(t)-sy(0)-y'(0)\}-\{sL[y(t)-y(0)\}-2L[y(t)]=0$$

Using initial conditions we have

$$Ly(t)=\frac{6}{s+2\quad s-1\quad s+1}$$

$$y(t)=L^{-1}\left\{\frac{6}{s+2\quad s-1\quad s+1}\right\}$$

$$\frac{6}{s+2\quad s-1\quad s+1}=\frac{A}{s+2}+\frac{B}{s-1}+\frac{C}{s+1}$$

solving we get $A = 2, B = 1, C = -3$

$$\therefore \frac{6}{(s+2)(s-1)(s+1)} = \frac{2}{s+2} + \frac{1}{s-1} + \frac{-3}{s+1}$$

$$L^{-1}\left\{\frac{6}{(s+2)(s-1)(s+1)}\right\} = 2L^{-1}\left(\frac{1}{s+2}\right) + L^{-1}\left(\frac{1}{s-1}\right) - 3L^{-1}\left(\frac{1}{s+1}\right)$$

$$y(t) = 2e^{-2t} + e^{t} - 3e^{-t}$$

7) Solve the initial value problem $(D^3 - 3D^2 + 3D - 1)y = 2t^2e^t$, $y(0) = 1, y'(0) = 0, y''(0) = -2$

Solution: Taking the laplace transform of the given equation, we get

$$s^3 y - s^2 y(0) - s y'(0) - y''(0) - 3\left[s^2 y - s y(0) - y'(0)\right] + 3\left[s y - y(0)\right] - y = 2.\frac{2}{(s-1)^3}$$

Using given conditions we get $(s^3 - 3s^2 + 3s - 1)y - (s^2 - 3s + 1) = \frac{4}{(s-1)^3}$

$$= (s-1)^3 y = (s^2 - 3s + 1) + \frac{4}{(s-1)^3}$$

$$\Rightarrow y = \frac{(s^2 - 3s + 1)}{(s-1)^3} + \frac{4}{(s-1)^6} = \frac{(s-1)^2 - (s-1) - 1}{(s-1)^3} + \frac{4}{(s-1)^6}$$

taking laplace transform we get

$$y = e^t\left\{1 - t - \frac{1}{2}t^2 + \frac{1}{30}t^5\right\}$$